

Length functions of Grothendieck categories with applications to infinite group representations

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Abstract

Let R be a ring, let G be a countable amenable group and let $R * G$ be a crossed product. The goal of this paper is to construct, starting with a suitable length function L on the category of left modules over R , a length function on a subcategory of the category of left modules over $R * G$, which coincides with the whole category if $L({}_R R) < \infty$. This construction can be performed using a dynamical invariant associated with the original length function L , called algebraic L -entropy. We apply our results to two classical problems on group rings: the Stable Finiteness and the Zero-Divisors Conjectures.

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1 Introduction

Consider a Grothendieck category \mathfrak{C} , for example, let \mathfrak{C} be the category $R\text{-Mod}$ of left R -modules over a ring R . In this paper we are interested in studying some particularly well-behaved numerical invariants of the objects of \mathfrak{C} called length functions. This notion was introduced by Northcott and Reufel [26] (in case $\mathfrak{C} = R\text{-Mod}$) in 1965 and it was also studied by Vámos [35, 36] and Zanardo [39]. Let us be more precise: a *length function* is an invariant $L : \mathfrak{C} \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$ such that

- (1) $L(0) = 0$ and $L(B) = L(A) + L(C)$ if $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is a short exact sequence, that is, L is *additive*;
- (2) $L(M) = \sup_I L(M_i)$, if M is the direct limit of a directed family $\{M_i : i \in I\}$ of subobjects, that is, L is *upper continuous*.

Classical examples of length functions of the category $R\text{-Mod}$ are the composition length $\ell(M)$ of a left R -module M , for R an arbitrary ring, the (torsion-free) rank $\text{rk}(M)$ of an arbitrary left R -module M over a left Ore domain R , and $\log |M|$ when $R = \mathbb{Z}$ and M is an Abelian group.

In Section 2, we give some generalities about torsion theories and localization in Grothendieck categories; using this formalism, we recall the definition of Gabriel dimension for Grothendieck categories proving some useful properties of this invariant. Gabriel dimension was introduced by Gabriel [14] under the name of Krull dimension; its name is due to Gordon and Robson [18].

One of the main results of Vámos' Ph.D. thesis [36] was to give a description of all the length functions of $R\text{-Mod}$ whenever this category has Gabriel dimension. Our Theorem 2.37 is an analogous statement for general Gabriel categories (i.e., Grothendieck categories with Gabriel dimension); in particular, we show that any length function in such a category is a linear combination of some "atomic length functions". The philosophical point is that any length function on a Gabriel category is a linear combination of functions which are induced by the composition length in a localization of the original category at some point of the Gabriel spectrum. We have made the choice to give complete proofs for various reasons. First of all, Vámos' thesis is not easily accessible as it was not published (excluding some particular cases [35]), furthermore it seemed natural to use the formalism of localization and torsion theories to give more direct arguments; I believe this approach clarifies some points of the theory. Nevertheless, many ideas and techniques come directly from [36].

Given a left Noetherian ring A with a distinguished central element $X \in A$, there is an important length function of the category of A -modules called *multiplicity* of X (see for example Chapter 7 of [25]). This invariant is based on the notion of composition length of modules. The classical notion of multiplicity was used by Vámos as a model to construct an " L -multiplicity of X " based on a given length function L of $A\text{-Mod}$ (see Chapter 5 of Vámos' thesis).

Let now R be a left Noetherian ring, let $A = R[X]$ be the ring of polynomials in one variable with coefficients in R , and let M be a left A -module. Given a length function $L : R\text{-Mod} \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$, one can extend trivially L to $A\text{-Mod}$ (just forgetting the action of X) and then take the L -multiplicity of the element $X \in A$, which we denote by mult_L . This defines a map

$$\begin{aligned} \text{mult} : \{\text{length functions on } R\text{-Mod}\} &\longrightarrow \{\text{length functions on } R[X]\text{-Mod}\} \\ L &\longmapsto \text{mult}_L. \end{aligned}$$

The values of L can be recovered via the formula

$$L(M) = \text{mult}_L(R[X] \otimes_R M) \quad \forall M \in R\text{-Mod}.$$

This procedure of extending length functions from a given ring to its ring of polynomials is useful in many situations but it has the disadvantage that it works just in the Noetherian case. In the recent paper [32], Salce, Vámos and the author studied the problem of the extension of a given length function

on a category of modules $R\text{-Mod}$ to a length function of (a suitable subcategories of) $R[X]\text{-Mod}$, without any hypothesis on the base ring. The key idea is to see a left $R[X]$ -module as a left R -module with a distinguished endomorphism, roughly speaking we see $R[X]$ -modules as discrete-time dynamical systems, and then we apply an invariant, called algebraic L -entropy, of the endomorphisms of left R -modules (see Subsection 5.3 for some more details and more references on these matters). It turns out that the algebraic L -entropy and the L -multiplicity coincide whenever they are both defined.

In this paper we study the following question, which is asking for a deep generalization of what we briefly explained above:

Question 1.1. *Let R be a ring, let G be a group and fix a crossed product $R * G$ (see Subsection 4.3). Is it possible to find a map*

$$\begin{aligned} \{\text{length functions on } R\text{-Mod}\} &\longrightarrow \{\text{length functions on } R * G\text{-Mod}\} \\ L &\longmapsto L_{R * G} \end{aligned}$$

*satisfying the formula $L(M) = L_{R * G}(R * G \otimes_R M)$ for any left R -module M ?*

We will give a partial answer to the above question using a notion of entropy similar to that used in [32]. More precisely, we will show that there are “many” length functions of $R\text{-Mod}$ which can be extended to a “large” subcategory of $R * G\text{-Mod}$ if we assume that the group G is amenable (see Subsection 4.1). Let us also remark that, even for $R = \mathbb{Z}$ and $L = \log_2 |\cdot|$, Elek [10] proved that it is not always possible to find a length function of the category $R[G]\text{-Mod}$ with the above properties (1)–(4) if G is not amenable. (The result of Elek is for compact Abelian groups but his statement implies what we said by Pontryagin-Van Kampen duality.) This shows that the answer to Question 1.1 is negative in general.

In the rest of the introduction we try to describe the main ideas of the paper and we expose our main results, including a couple of applications to classical problems. Indeed, let us start with a ring R , a left R -module ${}_R M$ and a group G . Analogously to the case of polynomial rings mentioned above, to specify a left $R[G]$ -module structure on ${}_R M$ we have just to choose a group homomorphism

$$\lambda : G \rightarrow \text{Aut}_R(M), \quad g \mapsto \lambda_g.$$

Looking things from a different angle, one can see the above map as a dynamical system where the ambient space is M , the time is the group G and the evolution law $T : G \times M \rightarrow M$ is such that $T(g, m) = \lambda_g(m)$. If one chooses to work with a general crossed product $R * G$ and a left $R * G$ -module M , then the analog of the above map λ induced by the $R * G$ -module structure of M is not a group homomorphism, so one cannot speak properly about an action of G . Anyway, G still acts on the monoid $\mathcal{L}(M)$ of R -submodules of M in the sense that there is a group homomorphism

$$\lambda' : G \rightarrow \text{Aut}(\mathcal{L}(M)), \quad g \mapsto \lambda'_g,$$

where $\text{Aut}(\mathcal{L}(M))$ is the group of monoid automorphisms of $\mathcal{L}(M)$. Consider now a length function $L : R\text{-Mod} \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$. In order to attach to λ' a well-behaved dynamical invariant based on L we need a compatibility condition, that is, we need that $L(N) = L(\lambda'_g(N))$ for all $g \in G$ and $N \in \mathcal{L}(M)$. In Section 3 we study this kind of compatibility conditions in a categorical setting.

In the first part of Section 4 we introduce the class of amenable groups. Then, inspired by the notion of amenable topological entropy introduced by Ornstein and Weiss [27], we associate to a length function $L : R\text{-Mod} \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$ satisfying suitable conditions a notion of *algebraic L -entropy* ent_L for left $R * G$ -modules. For doing so we need some results about quasi-tilings and non-negative real functions on the finite parts of an amenable group, that are recalled in Subsection 4.2.

More precisely, given a ring R (no hypothesis is needed on the ring) and a finitely generated amenable group G , fix a crossed product $R * G$ and a discrete length function $L : R\text{-Mod} \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$ (where discrete just means that the finite values of L form a subset of $\mathbb{R}_{\geq 0}$ which is order-isomorphic to \mathbb{N}) which is compatible with $R * G$ (see Definition 4.6). We define $\text{lFin}_L(R * G)$ to be the subclass of $R * G\text{-Mod}$ consisting of all left $R * G$ -modules ${}_R M$ such that $L(K) < \infty$, for any finitely generated R -submodule $K \in \mathcal{L}(M)$. For example, $\text{lFin}_L(R * G)$ contains all the left $R * G$ -modules ${}_R M$ such that $L(M) < \infty$. Furthermore, a consequence of the continuity of L on direct limits of submodules is that, given a left R -module ${}_R K$ such that $L(K) < \infty$, then the left $R * G$ -module $R * G \otimes_R K$ is in $\text{lFin}_L(R * G)$. The algebraic L -entropy is a function $\text{ent}_L : \text{lFin}_L(R * G) \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$. The main result of Section 6 is the following partial answer to Question 1.1:

Theorem 6.1 *In the above hypotheses, the invariant $\text{ent}_L : \text{lFin}_L(R * G) \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$ satisfies the following properties:*

- (1) ent_L is upper continuous;
- (2) $\text{ent}_L(R * G \otimes_R K) = L(K)$ for any L -finite left R -module K ;
- (3) $\text{ent}_L(R * G N) = 0$ if and only if $L(RN) = 0$, for any $R * G$ -submodule $N \leq R * G \otimes_R K$;
- (4) ent_L is additive.

In particular, ent_L is a length function of $\text{lFin}_L(R * G)$.

In Section 7 we present two possible applications of the machinery introduced in the present paper. In particular, we show that, making use of the algebraic L -entropy, one can prove a very strong form of the Kaplansky Stable Finiteness Conjecture, stating that group rings with coefficients in a field are stably finite (see Conjecture 7.2), for amenable groups. Recall that, given a ring R , a left R -module M is said to be *hopfian* if any endomorphism $\phi : M \rightarrow M$ is surjective if and only if it is bijective, while M is *hereditarily hopfian* if any of its submodules is hopfian; it is known that a ring R is stably finite if and only if the left R -modules of the form ${}_R R^n$ (with $n \in \mathbb{N}_+$) are hopfian. We prove the following

Theorem 7.8. *Let R be a left Noetherian ring, let G be a finitely generated amenable group and let $R * G$ be a fixed crossed product. Then, for any finitely generated left R -module ${}_R K \in R\text{-Mod}$, the left $R * G$ -module $R * G \otimes_R K$ is hereditarily hopfian.*

*In particular, $\text{End}_{R * G}(M)$ is stably finite for any submodule ${}_R M \leq R * G \otimes_R K$.*

The hopficity of all the submodules of $R[G] \otimes_R N$, with N a finitely generated left R -module, is a very strong property. In fact, we show in Example 7.11 that it does not hold when G is non-commutative free.

We conclude the paper showing, in Subsection 7.2, that a deeper understanding of the algebraic entropy would have implications also on the Zero-Divisor Conjecture, stating that the group algebras of torsion-free groups with coefficients in a field are domains. In fact, given a division ring \mathbb{K} and a finitely generated amenable group G , we show that a fixed crossed product $\mathbb{K} * G$ is a domain if and only if the algebraic entropy associated to the dimension of left \mathbb{K} -vector spaces takes values in $\mathbb{N} \cup \{\infty\}$ (see Theorem 7.14). In other words, the algebraic entropy detects the zero-divisors in $\mathbb{K} * G$. As an immediate consequence, we obtain that, in the above hypotheses, $\mathbb{K} * G$ is a domain if and only if it admits a flat embedding in a division ring.

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2 Torsion theories, length functions and Gabriel dimension

All along this section, when not differently specified, \mathfrak{C} denotes a Grothendieck category, that is, a cocomplete (i.e., Ab.3) Abelian category, with exact colimits (i.e., Ab.5) and with a generator. For unexplained terms and notations we refer to [34].

Remember that any Grothendieck category is well-powered (see [34, Proposition 6, p.94]) and that the set of all subobjects of a given object M is a bounded and bicomplete lattice, which we denote by $\mathcal{L}(M)$; the minimum of $\mathcal{L}(M)$ is the 0-object and the maximum is M itself. For every family $\{N_i : i \in I\}$ of subobjects of M we denote by $\sum_I N_i$ (resp., $\bigcap_I N_i$) the join (resp., meet) of the N_i in $\mathcal{L}(M)$. If $\{N_i : i \in I\}$ is a directed system of subobjects, $\sum_I N_i$ is called the *direct union* of the N_i and it is sometimes denoted also by $\bigcup_I N_i$. With this notation we can state the equivalent form of Grothendieck's axiom Ab.5 stating that, given an object M in \mathfrak{C} , a subobject K of M and a directed system $\{N_i : i \in I\}$ of subobjects of M ,

$$\left(\sum_I N_i \right) \cap K = \sum_I (N_i \cap K).$$

Let us recall the following fundamental properties of Grothendieck categories:

Fact 2.1. [34, Corollaries 4.3 and 4.4, pp.222–223] *Let \mathfrak{C} be a Grothendieck category, then*

- (1) \mathfrak{C} is complete;
- (2) \mathfrak{C} has enough injective objects.

Given a family $\{M_i : i \in I\}$ of objects of \mathfrak{C} , we denote by $\prod_I M_i$ and $\bigoplus_I M_i$ the product and the coproduct respectively. Given an object M of \mathfrak{C} , we denote by $E(M)$ the injective envelope of M . Given an endomorphism $\phi : M \rightarrow M$, a subobject N of M is ϕ -invariant if $\phi(N) \leq N$. Any subobject of a quotient (or, equivalently, a quotient of a subobject) of an object M is generically called a *segment* of M . A *series* of M is a finite chain of subobjects

$$\sigma : 0 = N_0 \leq N_1 \leq \dots \leq N_n = M. \quad (2.1)$$

The *factors* of σ are the segments of M of the form N_{i+1}/N_i for some $i < n$. If all the factors of σ are simple objects, then we say that σ is a *composition series* for M . If $\sigma' : 0 = N'_0 \leq N'_1 \leq \dots \leq N'_m = M$ is another series of M , we say that σ' is a refinement of σ if $\{N_0, \dots, N_n\} \subseteq \{N'_0, \dots, N'_m\}$. The following fact is well-known.

Fact 2.2 (Artin-Schreier's Refinement Theorem). *Let \mathfrak{C} be a Grothendieck category and let M be an object of \mathfrak{C} . If σ_1 and σ_2 are two series of M , then there exists a series σ of M that refines both σ_1 and σ_2 .*

2.1 Length functions

In any category \mathfrak{C} it is possible to define (real-valued) invariants to measure various finiteness properties of the objects. In general, we call *invariant* of \mathfrak{C} , any map $i : \mathfrak{C} \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$ such that $i(X) = i(X')$ whenever X and X' are isomorphic objects in \mathfrak{C} .

If we make some stronger assumption on the structure of the category \mathfrak{C} , we can refine our definition of invariant in order to obtain a more treatable notion. Indeed, suppose that \mathfrak{C} is an Abelian category (or, more generally, an exact category). The information that one usually wants to encode is about homological properties. In this setting it seems natural to ask that, given a short exact sequence

$$0 \rightarrow X_1 \rightarrow X_2 \rightarrow X_3 \rightarrow 0 \quad (2.2)$$

in \mathfrak{C} , we have $i(X_2) = i(X_1) + i(X_3)$. In this case, we say that i is *additive* on the sequence (2.2). If i is additive on all the short exact sequences of \mathfrak{C} and $i(0) = 0$, then we say that i is an *additive invariant* (or *additive function*).

In the following lemma we collect some useful properties of additive functions.

Lemma 2.3. *Let \mathfrak{C} be an Abelian category and let $i : \mathfrak{C} \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$ be an additive function. Then*

- (1) $i(X) \geq i(Y)$ for every segment Y of $X \in \mathfrak{C}$;
- (2) $i(X_1 + X_2) + i(X_1 \cap X_2) = i(X_1) + i(X_2)$ for every pair of subobjects X_1, X_2 of $X \in \mathfrak{C}$;
- (3) $\sum_{j \text{ odd}} i(X_j) = \sum_{j \text{ even}} i(X_j)$ for every exact sequence $0 \rightarrow X_1 \rightarrow X_2 \rightarrow \dots \rightarrow X_n \rightarrow 0$ in \mathfrak{C} .

A natural assumption in the context of Grothendieck categories is the upper continuity of invariants, given an object $X \in \mathfrak{C}$ and a directed set $\mathcal{S} = \{X_\alpha : \alpha \in \Lambda\}$ of subobjects of X such that $\sum_\Lambda X_\alpha = X$, we say that i is *continuous* on \mathcal{S} if

$$i(X) = \sup\{i(X_\alpha) : \alpha \in \Lambda\}. \quad (2.3)$$

If i is continuous on all the direct unions as above, we say that i is *upper continuous*. Upper continuity can be defined in arbitrary Abelian categories even if it seems more meaningful when direct limits exist and are exact.

Definition 2.4. *Let \mathfrak{C} be an Abelian category. An additive and upper continuous invariant $i : \mathfrak{C} \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$ is said to be a length function.*

In what follows we generally denote length functions by the symbol L . We remark that the usual definition of length function is given in module categories, which are in particular locally finitely generated Grothendieck categories. In this special setting, the usual definition of upper continuity is different (see part (3) of the following proposition). We now show that we are not defining a new notion of upper continuity but just generalizing this concept to arbitrary Grothendieck categories (similar observations, with analogous proofs, were already present in [36] for module categories).

Recall that, given an object M of a Grothendieck category \mathfrak{C} and an ordinal κ , a set $\{M_\alpha : \alpha < \kappa\}$ of subobjects of M is a *continuous chain* if

- (1) $M_\alpha \leq M_\beta$, provided $\alpha, \beta < \kappa$ and $\alpha \leq \beta$;
- (2) $\bigcup_{\alpha < \lambda} M_\alpha = M_\lambda$ for any limit ordinal $\lambda < \kappa$.

Proposition 2.5. *Let \mathfrak{C} be a Grothendieck category and $L : \mathfrak{C} \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$ be an additive function. Consider the following statements:*

- (1) *L is a length function;*
- (2) *given an object $M \in \mathfrak{C}$, an ordinal κ and a continuous chain $\{M_\alpha : \alpha < \kappa\}$ of subobjects of M such that $M = \bigcup_{\alpha < \kappa} M_\alpha$, we have that $L(M) = \sup\{L(M_\alpha) : \alpha < \kappa\}$;*
- (3) *for every object $M \in \mathfrak{C}$ we have that $L(M) = \sup\{L(F) : F \text{ finitely generated subobject of } M\}$.*

Then (1) \Leftrightarrow (2) and (2) \Leftrightarrow (3). If \mathfrak{C} is locally finitely generated, then the above statements are all equivalent.

Proof. (1) \Rightarrow (2) is trivial since continuous chains are directed sets. On the other hand, consider a directed set (I, \leq) and a directed system $\{M_i : i \in I\}$ of subobjects of M . If I is finite then I has a maximum, so there is nothing to prove. On the other hand, if I is an infinite set, one shows as in the proof of [17, Lemma 1.2.10], that (I, \leq) is the union of a continuous well-ordered chain of directed subsets, each of which has strictly smaller cardinality than I . One concludes by transfinite induction that (2) \Rightarrow (1).

Assume now (3) and consider a continuous chain as in part (2). For every finitely generated subobject F of M , there exists $\alpha < \kappa$ such that $F \leq M_\alpha$ and so we obtain that

$$L(M) = \sup\{L(F) : F \text{ f.g. subobject of } M\} \leq \sup\{L(M_\alpha) : \alpha < \kappa\} \leq L(M).$$

To conclude, notice that if \mathfrak{C} is locally finitely generated, then any object is the direct unions of the directed system of its finitely generated subobjects. In this situation, (1) clearly implies (3). \square

The notion of length function on a Grothendieck category is quite formal, this is why it seems useful to stop for a while and describe some concrete examples of length functions. We start with two easy ones.

Example 2.6. Let D be a left Ore domain, let $\Sigma = D \setminus \{0\}$ and let $Q = \Sigma^{-1}D$ be the ring of left fractions of D . For any $M \in D\text{-Mod}$, the torsion free rank $\text{rk}(M)$ is defined to be the dimension of the left module $\Sigma^{-1}M = Q \otimes_D M$ over the skew field Q . The torsion free rank $\text{rk} : D\text{-Mod} \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$, where $\text{rk}(M) = \infty$ whenever the rank of M is not finite, is a length function. In particular, the dimension of vector spaces over a field is a length function.

Example 2.7. The logarithm of the cardinality $\log |\cdot| : \mathbb{Z}\text{-Mod} \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$, where $\log |G| = \infty$ whenever G is not finite, is a length function.

In what follows we define the length of a lattice. This can be applied to the lattice of subobjects of a given object in a Grothendieck category obtaining the so-called composition length, which is the invariant inspiring the abstract notion of length function.

Definition 2.8. *Let $(\mathcal{L}, \leq, 0, 1)$ be a bounded lattice. A sequence of elements $\sigma : 0 = x_0 \leq x_1 \leq \dots \leq x_n = 1$ is said to be a series of \mathcal{L} . Furthermore, the number of strict inequalities in a series σ , is called the length of σ , in particular the above series has length n ; we write it $\ell(\sigma) = n$. The length of \mathcal{L} is*

$$\ell(\mathcal{L}) = \sup\{\ell(\sigma) : \sigma \text{ a series of } \mathcal{L}\}.$$

Example 2.9. Let \mathfrak{C} be a Grothendieck category. Notice that a series of $\mathcal{L}(M)$ is just a series of M . We can define a function $\ell : \mathfrak{C} \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$, $\ell(M) = \ell(\mathcal{L}(M))$. Clearly, $\ell(M) = 0$ if and only if $M = 0$, $\ell(M) = 1$ if and only if M is a simple object and $\ell(M) < \infty$ if and only if M is both Noetherian and Artinian. It is a standard result that ℓ is a length function (see for example [12] for a proof in module categories).

Other examples can be obtained lifting a known length function along a localization functor as shown in Section 2.4. Another strategy to produce new examples is that of “linearly combining” some known length functions. We now describe this technique.

Definition 2.10. (1) *Given a length function L of \mathfrak{C} and $\lambda \in \mathbb{R}_{\geq 0} \cup \{\infty\}$, we consider the function*

$$\lambda L : \mathfrak{C} \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\} \quad \text{such that} \quad \lambda L(M) = \lambda \cdot (L(M)), \quad \forall M \in \mathfrak{C},$$

with the convention that $\infty \cdot 0 = 0 \cdot \infty = 0$;

(2) given a set I and additive functions L_i of \mathfrak{C} for all $i \in I$, we consider the function

$$\sum_{i \in I} L_i : \mathfrak{C} \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\} \text{ such that } \sum_{i \in I} L_i(M) = \sup \left\{ \sum_{i \in F} L_i(M) : F \subseteq I \text{ finite} \right\}, \forall M \in \mathfrak{C}.$$

It is an easy exercise to prove that the sum of length functions and the multiplication of a length function by a constant are again length functions.

2.2 Localization and torsion theories in Grothendieck categories

In this subsection we summarize some of the classical results about localization in Grothendieck categories. Indeed, given a Grothendieck category \mathfrak{C} , we study the pairs (\mathbf{L}, η) , where $\mathbf{L} : \mathfrak{C} \rightarrow \mathfrak{C}$ is a (additive) functor and $\eta : id_{\mathfrak{C}} \rightarrow \mathbf{L}$ is a natural transformation such that $\mathbf{L}\eta : \mathbf{L} \rightarrow \mathbf{L}^2$ is a natural isomorphism and $\mathbf{L}\eta = \eta\mathbf{L}$. It is noticed in [21, Lemma 2.2] that the existence of such a pair is equivalent to the existence of an Abelian category \mathfrak{D} and an adjoint pair of functors $\mathbf{Q} : \mathfrak{C} \rightleftarrows \mathfrak{D} : \mathbf{S}$, where \mathbf{S} is fully faithful. In this situation $\mathbf{L} = \mathbf{S} \circ \mathbf{Q}$ and \mathfrak{D} is equivalent to $\mathbf{L}\mathfrak{C}$; furthermore, η is determined by \mathbf{L} (see [21, Remark 2.3]).

In what follows we focus on the case when \mathbf{L} is right exact, equivalently, \mathbf{Q} is exact. In such situation, one can prove that the category \mathfrak{D} above is again a Grothendieck category and that

$$\text{Ker}(\mathbf{L}) = \{X \in \mathfrak{C} : \mathbf{L}(X) = 0\} = \{X \in \mathfrak{C} : \mathbf{Q}(X) = 0\} = \text{Ker}(\mathbf{Q})$$

is what is usually called a hereditary torsion class. Recall that a subclass \mathcal{T} of a Grothendieck category \mathfrak{C} is a *hereditary torsion class* (or *localizing class*) if

(TC1) for every short exact sequence $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$ in \mathfrak{C} , we have that $Y \in \mathcal{T}$ if and only if $X, Z \in \mathcal{T}$;

(TC2) \mathcal{T} is closed under coproducts.

If \mathcal{T} satisfies (TC1) (but not necessarily (TC2)), then \mathcal{T} is called a *Serre class*. Dually, a class \mathcal{F} is a *hereditary torsion free class* if

(TF1) \mathcal{F} is closed under taking subobjects, extensions and injective envelopes;

(TF2) \mathcal{F} is closed under products.

To better understand localizations in Grothendieck categories, it is useful to introduce one more functor, that is, the right adjoint to the inclusion $\text{Ker}(\mathbf{L}) \rightarrow \mathfrak{C}$. This functor is usually called the *torsion functor* and it is associated with what is usually called a *hereditary torsion theory*. A *hereditary torsion theory* τ in \mathfrak{C} is a pair of classes $(\mathcal{T}, \mathcal{F})$ such that

(TT1) the class \mathcal{T} of τ -torsion objects is a hereditary torsion class;

(TT2) the class \mathcal{F} of τ -torsion free objects is a hereditary torsion free class;

(TT3) $\mathcal{T} = \{M \in \mathfrak{C} : \text{Hom}_{\mathfrak{C}}(M, F) = 0 \text{ for all } F \in \mathcal{F}\}$ and $\mathcal{F} = \{N \in \mathfrak{C} : \text{Hom}_{\mathfrak{C}}(T, N) = 0 \text{ for all } T \in \mathcal{T}\}$.

Since all the torsion theories in the sequel are hereditary, we just say “torsion theory” (resp., “torsion class” and “torsion free class”) to mean “hereditary torsion theory” (resp., “hereditary torsion class” and “hereditary torsion free class”).

Given a torsion theory $\tau = (\mathcal{T}, \mathcal{F})$ in \mathfrak{C} , one can construct the following localization sequence:

$$\mathcal{T} \begin{array}{c} \xrightarrow{\text{inc}} \\ \xleftarrow{\mathbf{T}_{\tau}} \end{array} \mathfrak{C} \begin{array}{c} \xleftarrow{\mathbf{Q}_{\tau}} \\ \xrightarrow{\mathbf{S}_{\tau}} \end{array} \mathfrak{C}/\mathcal{T}, \quad (2.4)$$

where both pairs of functors are adjunctions. In particular,

- inc is the inclusion of \mathcal{T} in \mathfrak{C} (and it is therefore fully faithful);
- \mathbf{T}_{τ} is an idempotent left-exact radical called the τ -torsion functor, which is constructed as follows: for every object $X \in \mathfrak{C}$, $\mathbf{T}_{\tau}(X)$ is the direct union of all the subobjects of X belonging to \mathcal{T} , while it is defined on maps by restriction. It is the right adjoint of inc ;
- the Grothendieck category \mathfrak{C}/\mathcal{T} , which is called the *localization of \mathfrak{C} at τ* , can be identified with the full subcategory of \mathfrak{C} given by the τ -local objects, that is, the objects $X \in \mathcal{F}$ such that $E(X)/X \in \mathcal{F}$. The functor \mathbf{S}_{τ} is called τ -section functor and it is the (fully faithful) inclusion of \mathfrak{C}/\mathcal{T} in \mathfrak{C} ;

- the exact τ -quotient functor \mathbf{Q}_τ can be constructed as follows: $\mathbf{Q}_\tau(X) \cong \pi^{-1}(\mathbf{T}_\tau(E(X')/X'))$, where $X' = X/\mathbf{T}_\tau(X)$ and $\pi : E(X') \rightarrow E(X')/X'$ is the canonical projection, for any object $X \in \mathfrak{C}$. The τ -quotient functor is left adjoint to the τ -section functor.

Notice that, $\text{Ker}(\mathbf{Q}_\tau) = \mathcal{T}$ and $\text{Im}(\mathbf{S}_\tau) \subseteq \mathcal{F}$.

Notice that there is a canonical way to associate a length function to any torsion theory:

Example 2.11. Let \mathfrak{C} be a Grothendieck category and let $\tau = (\mathcal{T}, \mathcal{F})$ be a torsion theory in \mathfrak{C} . Then the function $L : \mathfrak{C} \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$ such that

$$L(M) = \begin{cases} 0 & \text{if } M \in \mathcal{T}; \\ \infty & \text{if } M \notin \mathcal{T}; \end{cases}$$

is a length function.

2.3 Gabriel dimension and Gabriel categories

The *Gabriel filtration* of \mathfrak{C} is a transfinite chain $\{0\} = \mathfrak{C}_{-1} \subseteq \mathfrak{C}_0 \subseteq \dots \subseteq \mathfrak{C}_\alpha \subseteq \dots$ of torsion classes defined as follows:

- $\mathfrak{C}_{-1} = \{0\}$;
- suppose that α is an ordinal for which \mathfrak{C}_α has already been defined. An object $C \in \mathfrak{C}$ is said to be α -cocritical if C is τ_α -torsion free and every proper quotient of C is τ_α -torsion, where τ_α is the unique torsion theory whose torsion class is \mathfrak{C}_α . In the sequel, we will just say α -torsion (resp., torsion free) instead of τ_α -torsion (resp., torsion free). We let $\mathfrak{C}_{\alpha+1}$ be the smallest hereditary torsion class containing \mathfrak{C}_α and all the α -cocritical objects;
- if λ is a limit ordinal, we let \mathfrak{C}_λ be the smallest hereditary torsion class containing $\bigcup_{\alpha < \lambda} \mathfrak{C}_\alpha$.

An object M in \mathfrak{C} is said to be *cocritical* if it is α -cocritical for some α . For any ordinal α , we let $\mathbf{T}_\alpha : \mathfrak{C} \rightarrow \mathfrak{C}_\alpha$ and $\mathbf{Q}_\alpha : \mathfrak{C} \rightarrow \mathfrak{C}/\mathfrak{C}_\alpha$ be respectively the torsion and the localization functors. Abusing notation, we use the same symbols for the functors $\mathbf{T}_\alpha : \mathfrak{C}_{\alpha+1} \rightarrow \mathfrak{C}_\alpha$ and $\mathbf{Q}_\alpha : \mathfrak{C}_{\alpha+1} \rightarrow \mathfrak{C}_{\alpha+1}/\mathfrak{C}_\alpha$, induced by restriction.

Remark 2.12. By definition, a Grothendieck category \mathfrak{C} has a generator G . Furthermore, G has just a set (as opposed to a proper class) of subobject, equivalently, it has just a set of quotients. One can show that $\mathfrak{C}_{\alpha+1}$ is the smallest torsion class containing \mathfrak{C}_α and the α -cocritical quotients of G . As a consequence we obtain that there is an ordinal κ such that $\mathfrak{C}_\alpha = \mathfrak{C}_\kappa$ for all $\alpha \geq \kappa$ (just take $\kappa = \sup\{\alpha : \text{there are } \alpha\text{-cocritical quotients of } G\}$).

Consider the union $\bar{\mathfrak{C}} = \bigcup_\alpha \mathfrak{C}_\alpha$ of the Gabriel filtration (this makes sense by the above remark). An object belonging to $\bar{\mathfrak{C}}$ is said to be an object with Gabriel dimension. The Gabriel dimension of M is the minimal ordinal δ such that $M \in \mathfrak{C}_\delta$, in symbols $\text{G.dim}(M) = \delta$. In general, it may happen that $\mathfrak{C} \neq \bar{\mathfrak{C}}$; if $\mathfrak{C} = \bar{\mathfrak{C}}$, we say that \mathfrak{C} is a *Gabriel category* with *Gabriel dimension* $\text{G.dim}(\mathfrak{C}) = \kappa$, where κ is the smallest ordinal such that $\mathfrak{C}_\kappa = \mathfrak{C}$.

In the following example we specialize some of the above notions to categories of modules. Even if the example is stated for modules, part (1) holds for any Grothendieck category, taking a generator in place of ${}_R R$. Part (2) can be generalized to any Grothendieck category with a set of Noetherian generators.

Example 2.13. Let R be a ring and let $\mathfrak{C} = R\text{-Mod}$ be the category of left R -modules.

- (1) If \mathfrak{C} has Gabriel dimension $\text{G.dim}(\mathfrak{C}) = \kappa$, then $\text{G.dim}({}_R R) = \kappa$. In fact, the inequality $\text{G.dim}(\mathfrak{C}) \geq \text{G.dim}({}_R R)$ is trivial. For the converse inequality just notice that if α is an ordinal for which ${}_R R \in \mathfrak{C}_\alpha$, then $\mathfrak{C}_\alpha = \mathfrak{C}$. This is because \mathfrak{C}_α is closed under arbitrary direct sums and quotients, so it contains any quotient of a free module, that is, any module.
- (2) If R is left Noetherian then \mathfrak{C} is a Gabriel category. We have to prove that, given an ordinal α , $\mathfrak{C}_\alpha = \mathfrak{C}_{\alpha+1}$ if and only if ${}_R R \in \mathfrak{C}_\alpha$. But in fact, if α is an ordinal for which ${}_R R \notin \mathfrak{C}_\alpha$, then the set $\mathcal{I} = \{{}_R I \leq {}_R R : {}_R(R/I) \notin \mathfrak{C}_\alpha\}$ of all the left ideals I of R such that the left R -module R/I is not in \mathfrak{C}_α is non-empty (as it contains at least the trivial ideal). By the Noetherianity of R , \mathcal{I} has a maximal element \bar{I} . Then, ${}_R(R/\bar{I}) \notin \mathfrak{C}_\alpha$ but every proper quotient of such module is in \mathfrak{C}_α . So, ${}_R(R/\bar{I})$ is α -cocritical, in particular it belongs to $\mathfrak{C}_{\alpha+1}$, which therefore properly contains \mathfrak{C}_α .

In what follows we prove some (probably well-known) observations about Gabriel dimension.

Lemma 2.14. *Given a Gabriel category \mathfrak{C} , the following statements hold true:*

- (1) $\text{G.dim}(\mathfrak{C}) = \sup\{\text{G.dim}(M) : M \in \mathfrak{C}\};$
- (2) *let α be an ordinal $< \text{G.dim}(\mathfrak{C})$ and $M \in \mathfrak{C}$, then $M \in \mathfrak{C}_{\alpha+1}$ if and only if there exists an ordinal σ and a continuous chain $0 = N_0 \leq N_1 \leq \dots \leq N_\sigma = M$, such that N_{i+1}/N_i is either α -cocritical or α -torsion for every $i < \sigma$;*
- (3) *let λ be a limit ordinal $\leq \text{G.dim}(\mathfrak{C})$ and $M \in \mathfrak{C}$, then $M \in \mathfrak{C}_\lambda$ if and only if $M = \bigcup_{\alpha < \lambda} \mathbf{T}_\alpha(M)$;*
- (4) *if $N \leq M \in \mathfrak{C}$, then $\text{G.dim}(M) = \max\{\text{G.dim}(N), \text{G.dim}(M/N)\}$.*

Proof. (1) is clear.

(2) Let \mathcal{A} be the class of all objects which are union of a chain as in the statement. Since every hereditary torsion class is closed under taking direct limits and extension, we obtain the inclusion $\mathcal{A} \subseteq \mathfrak{C}_{\alpha+1}$. On the other hand, $\mathfrak{C}_{\alpha+1}$ is minimal between the hereditary torsion classes which contain \mathfrak{C}_α and the α -cocritical objects, thus the converse inclusion follows by the fact that \mathcal{A} is a hereditary torsion class.

(3) The proof is analogous to part (2) and follows by the fact that the class of all objects M which can be written as $M = \bigcup_{\alpha < \lambda} \mathbf{T}_\alpha(M)$ is a hereditary torsion class, which is contained in \mathfrak{C}_λ .

(4) Let $\text{G.dim}(M) = \alpha$, $\text{G.dim}(N) = \beta$ and $\text{G.dim}(M/N) = \gamma$. As $M \in \mathfrak{C}_\alpha$ and by the closure properties of \mathfrak{C}_α it is clear that also N and M/N belong to \mathfrak{C}_α . Hence $\alpha \geq \max\{\beta, \gamma\}$.

On the other hand, $\mathfrak{C}_\beta \cup \mathfrak{C}_\gamma \subseteq \mathfrak{C}_{\max\{\beta, \gamma\}}$ and so M is an extension of two objects of $\mathfrak{C}_{\max\{\beta, \gamma\}}$. As this class is closed under extensions, $M \in \mathfrak{C}_{\max\{\beta, \gamma\}}$, that is, $\alpha \leq \max\{\beta, \gamma\}$. \square

Recall that a Grothendieck category \mathfrak{C} is said to be *semi-Artinian* if $\text{G.dim}(\mathfrak{C}) = 0$. By part (2) of the above lemma, this is equivalent to say that every object in \mathfrak{C} is the union of a continuous chain of subobjects whose successive factors are simple objects. It is not difficult to show that this is equivalent to say that every object in \mathfrak{C} has a simple subobject.

Corollary 2.15. *Given a Gabriel category \mathfrak{C} , $\mathfrak{C}_{\alpha+1}/\mathfrak{C}_\alpha$ is semi-Artinian for all $\alpha < \text{G.dim}(\mathfrak{C})$.*

Proof. As we noticed, it is enough to show that, given $M \in \mathfrak{C}_{\alpha+1}$, $\mathbf{Q}_\alpha(M)$ has a simple subobject. By Lemma 2.14 (2), we can find a sequence $N_1 \leq N \leq M$, where $N_1 \in \mathfrak{C}_\alpha$ and N/N_1 is α -cocritical. By the exactness of \mathbf{Q}_α and the fact that $\mathbf{Q}_\alpha(N_1) = 0$, we have that $\mathbf{Q}_\alpha(N/N_1) \cong \mathbf{Q}_\alpha(N) \leq \mathbf{Q}_\alpha(M)$. It is an exercise to show that $\mathbf{Q}_\alpha(N)$ is simple. \square

Corollary 2.16. *Let \mathfrak{C} be a Gabriel category and $N \in \mathfrak{C}$ be a Noetherian object. Then, $\text{G.dim}(N)$ is a successor ordinal. Furthermore, there exists a finite series $0 = N_0 \leq N_1 \leq \dots \leq N_k = N$ such that N_i/N_{i-1} is cocritical for all $i = 1, \dots, k$.*

Proof. Suppose, looking for a contradiction, that $\text{G.dim}(N) = \lambda$ for some limit ordinal λ . By part (3) of the above lemma, $N = \bigcup_{\beta < \lambda} \mathbf{T}_\beta(N)$ and so, by the Noetherian condition, $N = \mathbf{T}_\beta(N) \in \mathfrak{C}_\beta$ for some $\beta < \lambda$, a contradiction. So we have that $\text{G.dim}(N) = \alpha + 1$ for some ordinal $\alpha < \text{G.dim}(\mathfrak{C})$. Let us proceed by transfinite induction. If $\alpha = -1$, then by part (2) of the above lemma, there exists a continuous chain $0 = N_0 \leq N_1 \leq \dots \leq N_\sigma = N$, such that N_{i+1}/N_i is either α -cocritical or α -torsion (i.e., trivial) for every $i < \sigma$. Such chain has to be finite by the Noetherian condition so we can conclude. Now, if $\alpha > -1$ (successor or limit ordinal), again by part (2) of the above lemma, there exists a continuous chain $0 = N_0 \leq N_1 \leq \dots \leq N_\sigma = N$, such that N_{i+1}/N_i is either α -cocritical or α -torsion for every $i < \sigma$. Such chain has to be finite by the Noetherian condition and it can be refined to a finite chain whose factors are all cocritical using the inductive hypothesis applied to each of the Noetherian α -torsion factors. \square

2.4 Operations on length functions

Let \mathfrak{C} be a Grothendieck category and let $\tau = (\mathcal{T}, \mathcal{F})$ be a torsion theory on \mathfrak{C} . In this section we are going to show how the length functions of \mathfrak{C} are related with the ones of \mathcal{T} and \mathfrak{C}/\mathcal{T} . We start with the following

Proposition 2.17. *In the above notation, there is a bijective correspondence*

$$f : \{\text{length functions } L \text{ of } \mathfrak{C} \text{ with } \mathcal{T} \subseteq \text{Ker}(L)\} \xleftrightarrow{\quad} \{\text{length functions of } \mathfrak{C}/\mathcal{T}\} : g.$$

Proof. The maps f and g are defined in Lemmas 2.18 and 2.19 respectively. It follows by the definitions that they are inverse one to the other. \square

Lemma 2.18. *If $L_\tau : \mathfrak{C}/\mathcal{T} \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$ is a length function, then there exists a unique length function $L : \mathfrak{C} \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$ such that $L(M) = L_\tau(\mathbf{Q}_\tau(M))$ for all $M \in \mathfrak{C}$. Furthermore, $\mathcal{T} \subseteq \text{Ker}(L)$. We set $g(L_\tau) = L$.*

Proof. Existence follows by the fact that \mathbf{Q}_τ is an exact functor that preserves colimits. Uniqueness is clear and the last statement comes from the fact that $\mathcal{T} = \text{Ker}(\mathbf{Q}_\tau)$. \square

Lemma 2.19. *If $L : \mathfrak{C} \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$ is a length function such that $\mathcal{T} \subseteq \text{Ker}(L)$, then there exists a unique length function $L_\tau : \mathfrak{C}/\mathcal{T} \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$ such that $L(M) = L_\tau(\mathbf{Q}_\tau(M))$ for all $M \in \mathfrak{C}$. We set $f(L) = L_\tau$.*

Proof. For every $M \in \mathfrak{C}$, there is an exact sequence of the form $0 \rightarrow T_1 \rightarrow M \rightarrow \mathbf{S}_\tau \mathbf{Q}_\tau(M) \rightarrow T_2 \rightarrow 0$, with $T_1, T_2 \in \mathcal{T}$. As by hypothesis L is trivial on τ -torsion objects, we obtain by additivity that $L(M) = L(\mathbf{S}_\tau \mathbf{Q}_\tau(M))$. Using this simple observation, we can define

$$L_\tau(N) = L(\mathbf{S}_\tau(N)), \quad \text{for all } N \in \mathfrak{C}/\mathcal{T},$$

and verify that $L(M) = L(\mathbf{S}_\tau \mathbf{Q}_\tau(M)) = L_\tau(\mathbf{Q}_\tau(M))$ as desired. The uniqueness statement follows by the fact that the functor \mathbf{Q}_τ is essentially surjective. It remains to verify that L_τ is a length function. Indeed, let $0 \rightarrow N \rightarrow M \rightarrow M/N \rightarrow 0$ be a short exact sequence in \mathfrak{C}/\mathcal{T} . This induces an exact sequence $0 \rightarrow \mathbf{S}_\tau(N) \rightarrow \mathbf{S}_\tau(M) \rightarrow \mathbf{S}_\tau(M/N) \rightarrow T \rightarrow 0$, with $T \in \mathcal{T}$. Hence, $L_\tau(M) = L(\mathbf{S}_\tau(M)) = L(\mathbf{S}_\tau(N)) + L(\mathbf{S}_\tau(M/N)) + 0 = L_\tau(N) + L_\tau(M/N)$. The proof that L is upper continuous follows by a similar argument and transfinite induction. \square

In the first part of this subsection we described how to transfer length functions along the adjoint pair $\mathbf{Q}_\tau : \mathfrak{C} \rightleftarrows \mathfrak{C}/\mathcal{T} : \mathbf{S}_\tau$; now we turn our attention to the adjunction $\text{inc} : \mathcal{T} \rightleftarrows \mathfrak{C} : \mathbf{T}_\tau$.

In particular, whenever L is a length function on \mathfrak{C} , one can define its *restriction* to \mathcal{T} as $L|_{\mathcal{T}} : \mathcal{T} \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$, such that $L|_{\mathcal{T}}(M) = L(M)$ for all $M \in \mathcal{T}$, and prove that it is a length function. Notice that this can be applied to any full Abelian subcategory \mathcal{T} of \mathfrak{C} , not only to hereditary torsion classes.

On the other hand, if we start with a length function L on \mathcal{T} , we want to find a canonical way to extend it to the bigger category \mathfrak{C} . In [36] (see also [35]) Peter Vámos introduced a technique to extend length functions which works in a more general setting. Indeed, let \mathcal{T} be a Serre subclass of \mathfrak{C} and consider a length function L on \mathcal{T} . We start defining an invariant (which is not supposed to have any good property but that of being useful for our constructions) $L^* : \mathfrak{C} \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$ as follows:

$$L^*(M) = \begin{cases} L(M) & \text{if } M \in \mathcal{T}; \\ 0 & \text{otherwise.} \end{cases} \quad (2.5)$$

Given an object $M \in \mathfrak{C}$ and a series $\sigma : 0 = N_0 \subseteq N_1 \subseteq \dots \subseteq N_n = M$ of M , we let

$$\hat{L}(\sigma) = \sum_{i \leq n} L^*(N_i/N_{i-1}).$$

Definition 2.20. *In the above notation, the Vámos extension $\hat{L} : \mathfrak{C} \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$ of L to \mathfrak{C} is the function defined by $\hat{L}(M) = \sup\{\hat{L}(\sigma) : \sigma \text{ ranging over all the series of } M\}$, for all $M \in \mathfrak{C}$.*

In the next proposition we verify that \hat{L} satisfies the axioms of a length function.

Proposition 2.21. *Let \mathcal{T} be a Serre subclass of a Grothendieck category \mathfrak{C} . If $L : \mathcal{T} \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$ is a length function, then $\hat{L} : \mathfrak{C} \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$ is a length function. Furthermore, if $L' : \mathfrak{C} \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$ is any additive function extending L , then $L'(M) \leq \hat{L}(M)$ for all $M \in \mathfrak{C}$.*

Proof. The proof of additivity is standard and it essentially follows by the definitions and the Artin-Schreier's Refinement Theorem. It remains to prove upper continuity. Indeed, let $M \in \mathfrak{C}$ and consider a directed set $\{M_\alpha : \alpha \in \Lambda\}$ of subobjects such that $\sum_\Lambda M_\alpha = M$. By additivity, we readily get $\hat{L}(M) \geq \sup_\Lambda \hat{L}(M_\alpha)$. On the other hand, given a series $\sigma : 0 = N_0 \subsetneq N_1 \subsetneq \dots \subsetneq N_n = M$, we prove by induction on $n \in \mathbb{N}_+$ that $\hat{L}(\sigma) \leq \sup_\Lambda \hat{L}(M_\alpha)$. We distinguish two cases:

- (1) if $\hat{L}(\sigma) = \infty$, then there exists a non-negative integer $m < n$ such that $N_{m+1}/N_m \in \mathcal{T}$ and $L(N_{m+1}/N_m) = \infty$. Notice also that $N_{m+1}/N_m = \sum_\Lambda ((M_\alpha \cap N_{m+1}) + N_m)/N_m$ and so,

$$\sup_\Lambda \hat{L}(M_\alpha) \geq \sup_\Lambda \hat{L}\left(\frac{(M_\alpha \cap N_{m+1}) + N_m}{N_m}\right) = \sup_\Lambda L\left(\frac{(M_\alpha \cap N_{m+1}) + N_m}{N_m}\right) = L(N_{m+1}/N_m) = \infty,$$

where the first inequality comes by additivity of \hat{L} and the following equalities come by the fact that \hat{L} coincides with L on \mathcal{T} .

- (2) Suppose now that $\widehat{L}(\sigma) < \infty$. If $n = 1$, then either $\widehat{L}(\sigma) = 0$ and there is nothing to prove, or $0 < \widehat{L}(\sigma) = L^*(M)$, but in this case $M \in \mathcal{T}$ and the thesis follows by the fact that L is a length function on \mathcal{T} . Suppose now $n > 1$, let

$$\sigma' : 0 = N_0 \subsetneq N_1 \subsetneq \cdots \subsetneq N_{n-1}, \text{ and } \sigma'' : 0 \subsetneq N_n/N_{n-1},$$

and notice that $\widehat{L}(\sigma) = \widehat{L}(\sigma') + \widehat{L}(\sigma'')$. Furthermore, $N_{n-1} = \sum_{\Lambda} (N_{n-1} \cap M_{\alpha})$ and $N_n/N_{n-1} = \sum_{\Lambda} (M_{\alpha} + N_{n-1})/N_{n-1}$. By inductive hypothesis $\widehat{L}(\sigma') \leq \sup_{\Lambda} \widehat{L}(N_{n-1} \cap M_{\alpha})$ and $\widehat{L}(\sigma'') \leq \sup_{\Lambda} \widehat{L}((M_{\alpha} + N_{n-1})/N_{n-1})$. Hence,

$$\widehat{L}(\sigma) = \widehat{L}(\sigma') + \widehat{L}(\sigma'') \leq \sup_{\Lambda} \widehat{L}(N_{n-1} \cap M_{\alpha}) + \sup_{\Lambda} \widehat{L}((M_{\alpha} + N_{n-1})/N_{n-1}) = \sup_{\Lambda} \widehat{L}(M_{\alpha}),$$

where the last equality comes from the additivity of \widehat{L} and the fact that the sum of suprema of two monotone nets is the supremum of the sum of the two nets. \square

We conclude this section proving that Vámos extension and lifting of length functions via a localization functor preserve linear combinations.

Lemma 2.22. *Let \mathcal{T} be a Serre subclass of the Grothendieck category \mathfrak{C} , let Λ be a set and choose $\lambda(\alpha) \in \mathbb{R}_{\geq 0} \cup \{\infty\}$ for all $\alpha \in \Lambda$.*

- (1) *Given length functions $L_{\alpha} : \mathcal{T} \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$ for all $\alpha \in \Lambda$ and letting $L = \sum_{\Lambda} \lambda(\alpha) L_{\alpha}$, we have that $\widehat{L} = \sum_{\Lambda} \lambda(\alpha) \widehat{L}_{\alpha}$.*
- (2) *If \mathcal{T} is closed under direct limits, $L_{\alpha} : \mathfrak{C}/\mathcal{T} \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$ are length functions for all $\alpha \in \Lambda$ and $L = \sum_{\Lambda} \lambda(\alpha) L_{\alpha}$, we have that $g(L) = \sum_{\Lambda} \lambda(\alpha) g(L_{\alpha})$, where g is the map of Lemma 2.18.*

Proof. (1) By the minimality of Vámos extension proved in Proposition 2.21, we have that $\widehat{L} \leq \sum_{\Lambda} \lambda(\alpha) \widehat{L}_{\alpha}$. On the other hand, if $M \in \mathfrak{C}$ and $\widehat{L}(M) = \infty$, then $\infty = \widehat{L}(M) \leq \sum_{\Lambda} \lambda(\alpha) \widehat{L}_{\alpha}(M)$ and there is nothing to prove. It remains to show that, if $\widehat{L}(M) < \infty$, then $\widehat{L}(M) \geq \sum_{\Lambda} \lambda(\alpha) \widehat{L}_{\alpha}(M)$. The case $|\Lambda| < \infty$ is essentially an application of the Artin-Schreier's Refinement Theorem. Suppose now that Λ is not finite and let $L_F = \sum_{\alpha \in F} \lambda(\alpha) L_{\alpha}$ for every non-empty finite subset $F \subseteq \Lambda$; by definition $L(M) = \sup\{L_F(M) : F \subseteq \Lambda \text{ finite}\}$. By the first part of the proof, $\widehat{L}_F = \sum_{\alpha \in F} \lambda(\alpha) \widehat{L}_{\alpha}$, so we have only to prove that $\widehat{L}(M) \geq \sup\{\widehat{L}_F(M) : F \subseteq \Lambda \text{ finite}\}$ for all $M \in \mathfrak{C}$. This follows noticing that $\widehat{L}_F(\sigma) \leq \widehat{L}(\sigma)$ for any finite $F \subseteq \Lambda$ and any series σ of M .

(2) follows by definition of the map g . \square

2.5 The classification in the semi-Artinian case

All along this subsection we denote by \mathfrak{C} a *semi-Artinian* Grothendieck category, that is, a Gabriel category whose Gabriel dimension is 0. The main result of this subsection is to give a structure theorem for all the length functions in \mathfrak{C} .

Lemma 2.23. *Let \mathfrak{C} be a semi-Artinian Grothendieck category and let $L, L' : \mathfrak{C} \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$ be two length functions. Then $L = L'$ if and only if their values on simple objects are the same.*

Proof. One implication is trivial, so suppose that L and L' coincide on simple objects. Consider an object $M \in \mathfrak{C}$ and write it as the union of a continuous chain

$$0 = N_0 \leq N_1 \leq \cdots \leq N_{\alpha} \leq \cdots \leq \bigcup_{\alpha} N_{\alpha} = M,$$

such that N_{i+1}/N_i is a simple object for all i (see Lemma 2.14(2)). By hypothesis $L(N_{i+1}/N_i) = L'(N_{i+1}/N_i)$ for all i . The conclusion follows by transfinite induction using additivity and upper continuity. \square

Definition 2.24. *Given a semi-Artinian category \mathfrak{C} , we let $\text{Sp}(\mathfrak{C})$ be the set of (isomorphism classes of) all the simple objects in \mathfrak{C} .*

In a general Abelian category, the *Gabriel spectrum* is the set of (isomorphism classes of) all the indecomposable injective objects. One can prove that, for a semi-Artinian Grothendieck category \mathfrak{C} , $\text{Sp}(\mathfrak{C})$ is in bijection with the Gabriel spectrum. In fact, the indecomposable injective objects in \mathfrak{C} are exactly the injective envelopes of simple objects.

Let $S \in \text{Sp}(\mathfrak{C})$ be a simple object. There is a torsion theory $\pi = (\mathcal{T}_\pi, \mathcal{F}_\pi)$ such that

$$\mathcal{T}_\pi = \{M \in \mathfrak{C} : \text{Hom}_{\mathfrak{C}}(M, E(S)) = 0\}$$

(and \mathcal{F}_π is the maximal right Hom-orthogonal class to \mathcal{T}_π). One can show that π determines S up to isomorphism, in fact, S is the unique simple object (up to isomorphism) in \mathcal{F}_π ; for this reason, we often abuse notation and write $\pi \in \text{Sp}(\mathfrak{C})$ and denote by $S(\pi)$ the simple object associated to the torsion theory π .

Definition 2.25. Let \mathfrak{C} be a semi-Artinian Grothendieck category and let $\pi \in \text{Sp}(\mathfrak{C})$. The atomic length function ℓ_π associated to π is the lifting (in the sense of Lemma 2.18) of the composition length $\ell : \mathfrak{C}/\mathcal{T}_\pi \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$ via the localization functor $\mathbf{Q}_\pi : \mathfrak{C} \rightarrow \mathfrak{C}/\mathcal{T}_\pi$, that is, $\ell_\pi : \mathfrak{C} \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$ is the unique length function such that

$$\ell_\pi(M) = \ell(\mathbf{Q}_\pi(M)), \quad \forall M \in \mathfrak{C}.$$

By Lemma 2.18, $\text{Ker}(\ell_\pi) = \mathcal{T}_\pi$ and, given a simple object $S \in \mathfrak{C}$, one easily shows that

$$\ell_\pi(S) = \begin{cases} 1 & \text{if } S \cong S(\pi); \\ \ell(Q_\pi(S)) & \text{otherwise.} \end{cases}$$

In the following theorem we prove that any length function in a semi-Artinian Grothendieck category is a linear combination of atomic length functions.

Theorem 2.26. Let \mathfrak{C} be a semi-Artinian category and let $L : \mathfrak{C} \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$ be a length function. Then

$$L = \sum_{\pi \in \text{Sp}(\mathfrak{C})} \lambda(\pi) \cdot \ell_\pi,$$

where $\lambda(\pi) = L(S(\pi))$. Furthermore, the constants $\lambda(\pi)$ are uniquely determined by L .

Proof. Let $L' = \sum_{\pi \in \text{Sp}(\mathfrak{C})} \lambda(\pi) \cdot \ell_\pi$; we already mentioned that a linear combination of length functions is a length function, so L' is a length function. Also, we already noticed that $\ell_\pi(S(\pi')) = 0$ for all $\pi' \neq \pi$, so $L'(S(\pi)) = \lambda(\pi) \ell_\pi(S(\pi)) = L(S(\pi))$ for all $\pi \in \text{Sp}(\mathfrak{C})$, which shows that $L = L'$, by Lemma 2.23. The proof of the uniqueness statement is analogous. \square

Using the uniqueness of the above decomposition, we can unambiguously define the *support* of L as

$$\text{Supp}(L) = \{\pi \in \text{Sp}(\mathfrak{C}) : \lambda(\pi) \neq 0\}.$$

As an immediate consequences of the above theorem we obtain the following corollaries.

Corollary 2.27. Let \mathfrak{C} be a semi-Artinian and local (i.e., with a unique simple object) Grothendieck category. Any length function $L : \mathfrak{C} \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$ is multiple of the composition length.

Corollary 2.28. Let D be a left Ore domain and denote by $\mathcal{T} \subseteq D\text{-Mod}$ the class of torsion left D -modules. The following are equivalent for a length function $L : D\text{-Mod} \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$:

- (1) $\mathcal{T} \subseteq \text{Ker}(L)$;
- (2) there exists $\alpha \in \mathbb{R}_{\geq 0} \cup \{\infty\}$ such that $L = \alpha \cdot \text{rk}$ (see Example 2.6).

Finally, if $L(D) < \infty$ then the above equivalent conditions are verified with $\alpha = L(D)$.

Proof. The implication (2) \Rightarrow (1) is trivial, in fact $\text{Ker}(\text{rk}) = \mathcal{T}$. Let us prove that (1) \Rightarrow (2). By Theorem 2.17, L is the lifting of a length function on $D\text{-Mod}/\mathcal{T} \cong Q\text{-Mod}$, where Q is the skew field of left fractions of D . Clearly, $Q\text{-Mod}$ satisfies the hypotheses of the above corollary, thus there exists $\alpha \in \mathbb{R}_{\geq 0} \cup \{\infty\}$ such that $L = g(\alpha \cdot \dim_Q) = \alpha \cdot g(\dim_Q) = \alpha \cdot \text{rk}$.

For the last statement, we suppose $L(D) < \infty$ and we verify (1). Let M be a torsion left D -module. Using upper continuity and additivity we can prove that $L(M) = 0$ if and only if $L(Dx) = 0$ for all $x \in M$. Thus, let $x \in M$ be a non-trivial element and consider the following exact sequences

$$0 \rightarrow \text{Ann}_D(x) \rightarrow D \rightarrow Dx \rightarrow 0 \quad \text{and} \quad 0 \rightarrow D \rightarrow \text{Ann}_D(x),$$

where the second sequence exists as $\text{Ann}_D(x)$ is a non-trivial left ideal of D (as M is torsion) and D is a domain. This shows that $L(D) = L(Dx) + L(\text{Ann}_D(x)) \geq L(Dx) + L(D)$; thus $L(Dx) \leq L(D) - L(D) = 0$. Finally, $L(D) = \alpha \cdot \text{rk}(D) = \alpha$. \square

2.6 The main structure theorem

All along this section we let \mathfrak{C} be a Gabriel category such that $\text{G.dim}(\mathfrak{C}) = \kappa$. The main result of this subsection is to show that, analogously to the semi-Artinian case, any length function on \mathfrak{C} can be written as a linear combination of atomic length functions.

Let us start defining atomic length functions in this broader context. Indeed, given an ordinal $\alpha \leq \kappa$ and letting \mathfrak{C}_α be the α -th member of the Gabriel filtration of \mathfrak{C} , we already noticed that all the localized categories of the form $\mathfrak{C}_{\alpha+1}/\mathfrak{C}_\alpha$, are semi-Artinian.

Definition 2.29. *Let \mathfrak{C} be a Gabriel category with $\text{G.dim}(\mathfrak{C}) = \kappa$. We let $\text{Sp}^\alpha(\mathfrak{C}) = \text{Sp}(\mathfrak{C}_{\alpha+1}/\mathfrak{C}_\alpha)$ and $\text{Sp}(\mathfrak{C}) = \bigcup_{\alpha < \kappa} \text{Sp}^\alpha(\mathfrak{C})$.*

For $\kappa = 0$, the above definition of $\text{Sp}(\mathfrak{C})$ coincides with our previous definition. As for semi-Artinian categories, one can find a natural bijection between $\text{Sp}(\mathfrak{C})$ and the Gabriel spectrum of \mathfrak{C} . In fact, any indecomposable injective object in \mathfrak{C} is isomorphic to $E(\mathbf{S}_\alpha(C))$ for some $\alpha > \kappa$ and some simple object $C \in \text{Sp}^\alpha(\mathfrak{C})$.

For any $\alpha < \kappa$ and $\pi \in \text{Sp}^\alpha(\mathfrak{C})$, we already defined a length function (see Definition 2.25)

$$\ell_\pi : \mathfrak{C}_{\alpha+1}/\mathfrak{C}_\alpha \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}.$$

Definition 2.30. *Let \mathfrak{C} be a Gabriel category and let $\alpha < \text{G.dim}(\mathfrak{C})$. For any given $\pi \in \text{Sp}^\alpha(\mathfrak{C})$, the atomic length function $\ell_\pi : \mathfrak{C} \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$ is defined as the Vámos extension of the lifting of the atomic length function $\ell_\pi : \mathfrak{C}_{\alpha+1}/\mathfrak{C}_\alpha \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$ to a length function on $\mathfrak{C}_{\alpha+1}$ (in the sense of Lemma 2.18).*

It follows directly by the definitions that $\mathfrak{C}_\alpha \subseteq \text{Ker}(\ell_\pi)$.

For the rest of this subsection we fix a length function L on \mathfrak{C} . We also let $\text{Fin}(L)$ be the Serre class of all the L -finite objects,

$$\text{Fin}(L) = \{M \in \mathfrak{C} : L(M) < \infty\}.$$

Furthermore, we denote by $\overline{\text{Fin}}(L)$ the minimal torsion class containing $\text{Fin}(L)$.

Definition 2.31. *The finite component L^{fin} of L is the Vámos extension to \mathfrak{C} of the restriction of L to $\text{Fin}(L)$, that is $L^{fin} = L|_{\text{Fin}(L)} : \mathfrak{C} \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$.*

The infinite component L^∞ of L is defined by

$$L^\infty(M) = \begin{cases} 0 & \text{if } M \in \overline{\text{Fin}}(L); \\ \infty & \text{otherwise.} \end{cases}$$

Notice that the length function L^∞ belongs to the family of functions described in Example 2.11. The fact that L^{fin} is a length function follows by Proposition 2.21. We remark that the finite component L^{fin} can very well assume infinite values (if L^{fin} is non-trivial, just take M such that $L^{fin}(M) \neq 0$, so $L^{fin}(\bigoplus_{\mathbb{N}} M) = \infty$); anyway its name is justified by the fact that L^{fin} is, by definition, determined by the finite values of L .

Notice that $L = L^{fin} + L^\infty$. This decomposition of L allows us to reduce the problem of finding a presentation of L as linear combination of atomic length functions to the same problem for L^∞ and L^{fin} . On the other hand, the decomposition of L^∞ is quite easy. Indeed, for all $\alpha < \kappa$ we let $\tau_\alpha^L = (\mathcal{T}_\alpha^L, \mathcal{F}_\alpha^L)$ be the torsion theory in $\mathfrak{C}_{\alpha+1}/\mathfrak{C}_\alpha$ such that $\mathcal{T}_\alpha^L = \{\mathbf{Q}_\alpha(M) : M \in \overline{\text{Fin}}(L)\}$.

Definition 2.32. *In the above notation, the infinite support of L is the following subset of $\text{Sp}(\mathfrak{C})$:*

$$\text{Supp}^\infty(L) = \bigcup_{\alpha < \kappa} \text{Supp}_\alpha^\infty(L), \quad \text{where} \quad \text{Supp}_\alpha^\infty(L) = \{\pi \in \text{Sp}^\alpha(\mathfrak{C}) : S(\pi) \in \mathcal{F}_\alpha^L\}.$$

Notice that $\mathcal{T}_\alpha^L = \bigcap_{\pi \in \text{Supp}_\alpha^\infty(L)} \mathcal{T}_\pi = \bigcap_{\pi \in \text{Supp}_\alpha^\infty(L)} \text{Ker}(\ell_\pi)$, in $\mathfrak{C}_{\alpha+1}/\mathfrak{C}_\alpha$.

Proposition 2.33. *In the above notation, $L^\infty = \sum_{\pi \in \text{Supp}^\infty(L)} \infty \cdot \ell_\pi$.*

Proof. Let $L' = \sum_{\pi \in \text{Supp}^\infty(L)} \infty \cdot \ell_\pi$. Both L^∞ and L' take values in $\{0, \infty\}$, thus they coincide if and only if $\text{Ker}(L^\infty) = \text{Ker}(L')$. By the above observation,

$$\text{Ker}(L^\infty) = \overline{\text{Fin}}(L) = \bigcap_{\alpha < \kappa} \{M \in \mathfrak{C} : \mathbf{Q}_\alpha \mathbf{T}_{\alpha+1}(M) \in \mathcal{T}_\alpha^L\} = \bigcap_{\pi \in \text{Supp}^\infty(L)} \text{Ker}(\ell_\pi) = \text{Ker}(L').$$

□

We can now turn our attention to the decomposition of L^{fin} .

Lemma 2.34. *In the above notation, there exists a unique family $\{L_{(\alpha)} : \mathfrak{C}_{\alpha+1} \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\} : \alpha < \kappa\}$ of length functions such that*

- (1) $L^{fin} = \sum_{\alpha < \kappa} L_{(\alpha)}$, where $L_{(\alpha)} : \mathfrak{C} \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$ is the Vámos extension of $L_{(\alpha)}$;
- (2) $\mathfrak{C}_{\alpha} \subseteq \text{Ker}(L_{(\alpha)})$, for all $\alpha < \kappa$.

Proof. For all $\alpha \leq \text{G.dim}(\mathfrak{C})$ we consider the Serre classes $\text{Fin}^{(\alpha)}(L) = \text{Fin}(L) \cap \mathfrak{C}_{\alpha}$ and $\text{Fin}(L)$. We start defining inductively length functions $L^{(\alpha)} : \text{Fin}^{(\alpha+1)}(L) \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$, and their Vámos extensions $L^{\alpha} : \text{Fin}(L) \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$, for all $\alpha < \text{G.dim}(\mathfrak{C})$:

- $L^{(-1)}(M) = L(M)$, for all $M \in \text{Fin}^{(0)}(L)$;
- $L^{(\alpha)}(M) = L(M) - \sum_{\beta < \alpha} L^{\beta}(M)$, for all $M \in \text{Fin}^{(\alpha+1)}(L)$ and $\alpha < \text{G.dim}(\mathfrak{C})$.

It is not difficult to verify by transfinite induction that all the $L^{(\alpha)}$ and L^{α} are length functions. Let us verify the following claims by induction on $\text{G.dim}(\mathfrak{C})$:

- (1') $L(M) = \sum_{\alpha < \text{G.dim}(\mathfrak{C})} L^{\alpha}(M)$, for all $M \in \text{Fin}(L)$;
- (2') $\text{Fin}^{(\alpha)}(L) \subseteq \text{Ker}(L^{(\alpha)})$, for all $\alpha < \text{G.dim}(\mathfrak{C})$.

If $\text{G.dim}(\mathfrak{C}) = 0$ (i.e., \mathfrak{C} is semi-Artinian), then $L(M) = L^{(-1)}(M) = L^{-1}(M)$ for all $M \in \text{Fin}^{(0)}(L) = \text{Fin}(L)$, proving (1'), while (2') is trivial since $\text{Fin}^{(-1)}(L) = \{0\}$.

Suppose now that $\text{G.dim}(\mathfrak{C})$ is a limit ordinal. If $M \in \text{Fin}^{(\alpha)}(L)$ for some $\alpha < \text{G.dim}(\mathfrak{C})$, then by inductive hypothesis, $L(M) = \sum_{\beta < \alpha} L^{\beta}(M)$ and so $L^{(\alpha)}(M) = L(M) - \sum_{\beta < \alpha} L^{\beta}(M) = 0$, proving (2'). Furthermore, given $M \in \text{Fin}(L)$, we can write $M = \bigcup_{\beta < \text{G.dim}(\mathfrak{C})} (\mathbf{T}_{\beta}(M))$ (see Lemma 2.14 (3)). Then,

$$\begin{aligned} L(M) &= \sup_{\beta < \text{G.dim}(\mathfrak{C})} L(\mathbf{T}_{\beta}(M)) = \sup_{\beta < \text{G.dim}(\mathfrak{C})} \left\{ \sum_{\alpha \leq \beta} L^{\alpha}(\mathbf{T}_{\beta}(M)) \right\} \\ &= \sup_{\beta < \text{G.dim}(\mathfrak{C})} \left\{ \sum_{\alpha < \text{G.dim}(\mathfrak{C})} L^{\alpha}(\mathbf{T}_{\beta}(M)) \right\} = \sum_{\alpha < \text{G.dim}(\mathfrak{C})} L^{\alpha}(M), \end{aligned}$$

where the first equality follows by the upper continuity of L , the second one follows by part (1') of the inductive hypothesis ($\mathbf{T}_{\beta}(M) \in \mathfrak{C}_{\beta}$ and $\text{G.dim}(\mathfrak{C}_{\beta}) = \beta < \text{G.dim}(\mathfrak{C})$), the third one follows by the, already established claim (2'), and the last equality uses the upper continuity of $\sum_{\alpha < \text{G.dim}(\mathfrak{C})} L^{\alpha}$.

Finally, if $\text{G.dim}(\mathfrak{C}) = \kappa + 1$ is a successor ordinal, and $M \in \text{Fin}^{(\kappa)}(L)$, then $L(M) = \sum_{\alpha < \kappa} L^{\alpha}(M)$ by inductive hypothesis and so $L^{(\kappa)}(M) = L(M) - \sum_{\alpha < \kappa} L^{\alpha}(M) = 0$, proving (2'). Furthermore, for any $M \in \text{Fin}(L)$,

$$\sum_{\alpha \leq \kappa} L^{\alpha}(M) = \sum_{\alpha < \kappa} L^{\alpha}(M) + L^{\kappa}(M) \stackrel{(*)}{=} \sum_{\alpha < \kappa} L^{\alpha}(M) + L(M) - \sum_{\alpha < \kappa} L^{\alpha}(M) = L(M),$$

where (*) comes by the fact that $\mathfrak{C}_{\kappa+1} = \mathfrak{C}$ so $\text{Fin}^{(\kappa+1)}(L) = \text{Fin}(L)$ and $L^{\kappa} = L^{(\kappa)}$. Thus, also (1') is established.

For any $\alpha < \text{G.dim}(\mathfrak{C})$ we define the length functions $L_{(\alpha)} : \mathfrak{C}_{\alpha+1} \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$ and $L_{\alpha} : \mathfrak{C} \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$, as the Vámos extensions of $L^{(\alpha)}$ and L^{α} respectively. We can extend the above claims (1') and (2') to these new functions as follows. First of all, $L^{fin}(M) = \sum_{\alpha < \text{G.dim}(\mathfrak{C})} L_{\alpha}(M)$ by (1') and the fact that Vámos extension preserves linear combinations. Furthermore, for all $\alpha < \text{G.dim}(\mathfrak{C})$ and $M \in \mathfrak{C}_{\alpha}$, $L_{(\alpha)}(M) = 0$ by the construction of Vámos extension and since, by (2'), $L^{(\alpha)}$ vanishes on all the factors belonging $\text{Fin}^{(\alpha)}(\mathfrak{C})$ of any series of M . This shows that $\mathfrak{C}_{\alpha} \subseteq \text{Ker}(L_{(\alpha)})$.

The proof of the uniqueness can be obtained by transfinite induction on κ . □

By the above lemma, we have uniquely determined length functions $L_{(\alpha)} : \mathfrak{C}_{\alpha+1} \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$ for all $\alpha < \text{G.dim}(\mathfrak{C})$, such that $L_{(\alpha)}$ is trivial on \mathfrak{C}_{α} . Thus, there exist unique length functions

$$\overline{L}_{\alpha} : \mathfrak{C}_{\alpha+1}/\mathfrak{C}_{\alpha} \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\} \quad (2.6)$$

such that $L_{(\alpha)}(M) = \overline{L}_{\alpha}(\mathbf{Q}_{\alpha}(M))$ for all $M \in \mathfrak{C}_{\alpha+1}$ (see Lemma 2.19). Since all the categories of the form $\mathfrak{C}_{\alpha+1}/\mathfrak{C}_{\alpha}$ are semi-Artinian and so we have a well defined notion of support for the functions \overline{L}_{α} .

Definition 2.35. In the above notation, the finite support of L is the following subset of $\text{Sp}(\mathfrak{C})$:

$$\text{Supp}^{fin}(L) = \bigcup_{\alpha < \kappa} \text{Supp}(\overline{L}_\alpha).$$

With the above notion of support we can finally decompose L^{fin} as linear combination of atomic length functions.

Proposition 2.36. In the above notation, there is a unique choice of constants $\lambda(\pi) \in \mathbb{R}_{>0}$ so that

$$L^{fin} = \sum_{\pi \in \text{Supp}^{fin}(L)} \lambda(\pi) \cdot \ell_\pi.$$

Proof. For all $\alpha < \kappa$, we have a uniquely determined decomposition $\overline{L}_\alpha = \sum_{\pi \in \text{Supp}(\overline{L}_\alpha)} \lambda(\pi) \ell_\pi$ in $\mathfrak{C}_{\alpha+1}/\mathfrak{C}_\alpha$, by Theorem 2.26. Furthermore, by Lemma 2.22, this decomposition can be lifted to a decomposition in $\mathfrak{C}_{\alpha+1}$ and then extended to a decomposition in \mathfrak{C} , obtaining that

$$L_\alpha = \sum_{\pi \in \text{Supp}(\overline{L}_\alpha)} \lambda(\pi) \ell_\pi, \quad \text{for all } \alpha < \kappa.$$

The desired decomposition now follows by Lemma 2.34. \square

We conclude this section summarizing the main results on decomposition of length functions in Gabriel categories in the following theorem. We remark that this statement is analogous to the “Main Decomposition Theorem” in [36]

Theorem 2.37. Let \mathfrak{C} be a Gabriel category and let $L : \mathfrak{C} \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$ be a length function. Then, there is a unique way to choose constants $\lambda(\pi) \in \mathbb{R}_{>0}$, for all $\pi \in \text{Supp}^{fin}(L)$ such that

$$L = L^{fin} + L^\infty = \sum_{\pi \in \text{Supp}^{fin}(L)} \lambda(\pi) \cdot \ell_\pi + \sum_{\pi \in \text{Supp}^\infty(L)} \infty \cdot \ell_\pi.$$

3 Length functions compatible with self-equivalences

All along this section, let \mathfrak{C} be a Grothendieck category, let $L : \mathfrak{C} \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$ be a length function and let $F : \mathfrak{C} \rightarrow \mathfrak{C}$ be a self-equivalence, that is,

- (Eq. 1) F is *essentially surjective*, i.e., for all $X \in \mathfrak{C}$, there exists $Y \in \mathfrak{C}$ such that $F(X) \cong Y$;
- (Eq. 2) F is *fully faithful*, i.e., for all $X, Y \in \mathfrak{C}$, the natural morphism $\text{Hom}_{\mathfrak{C}}(X, Y) \rightarrow \text{Hom}_{\mathfrak{C}}(F(X), F(Y))$ is an isomorphism.

A consequence of the definition is that F preserves any structure defined by universal properties, in particular, F commutes with direct and inverse limits and it preserves exactness of sequences. Furthermore, F commutes with injective envelopes and it preserves lattices of subobjects.

It is easily seen that

$$L_F : \mathfrak{C} \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\} \quad \text{such that} \quad L_F(M) = L(F(M)), \quad (3.1)$$

for all $M \in \mathfrak{C}$ is a length function. In what follows we are going to study to what extent L_F can differ from L .

The following example shows that L and L_F may be very different.

Example 3.1. Consider a field K and consider the category $K \times K\text{-Mod} \cong K\text{-Mod} \times K\text{-Mod}$. This category is semi-Artinian and it has a self-equivalence $F : K \times K\text{-Mod} \rightarrow K \times K\text{-Mod}$ such that $(M, N) \mapsto (N, M)$ and $(\phi, \psi) \mapsto (\psi, \phi)$. If we take L to be the length function such that $L((M, N)) = \dim_K(M)$, then clearly $L_F((M, 0)) = 0 \neq \dim_K(M) = L((M, 0))$, provided $M \neq 0$.

Definition 3.2. Given a Grothendieck category \mathfrak{C} and a self-equivalence $F : \mathfrak{C} \rightarrow \mathfrak{C}$, we say that a length function $L : \mathfrak{C} \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$ is compatible with F provided $L_F(M) = L(F(M)) = L(M)$ for all $M \in \mathfrak{C}$.

In this section we exploit the classification of length functions in Gabriel categories to find a necessary and sufficient condition on L to be compatible with F . Our motivation for studying compatibility of length functions with self-equivalences is the following: given a ring R and a ring automorphism $\phi : R \rightarrow R$, we define a self-equivalence

$$F_\phi : R\text{-Mod} \rightarrow R\text{-Mod}, \quad (3.2)$$

which is the scalar restriction along ϕ . That is, take a left R -module ${}_R M$ such that R acts on M via a ring homomorphism $\lambda : R \rightarrow \text{End}_{\mathbb{Z}}(M)$. Then $F_\phi(M)$ is isomorphic to M as an Abelian group, while R acts on $F_\phi(M)$ via the ring homomorphism $\lambda \circ \phi : R \rightarrow \text{End}_{\mathbb{Z}}(F_\phi(M))$.

We are interested in finding length functions L such that $L(F_\phi(M)) = L(M)$.

3.1 Orbit-decomposition of the Gabriel spectrum

We start with a technical result. Let \mathcal{A} be a subclass of \mathfrak{C} , we denote by $\tilde{\mathcal{A}}$ the class of all the objects of \mathfrak{C} which are isomorphic to some object in \mathcal{A} .

Lemma 3.3. *Let \mathfrak{C} be a Grothendieck category, let $F : \mathfrak{C} \rightarrow \mathfrak{C}$ a self-equivalence and let $\tau = (\mathcal{T}, \mathcal{F})$ be a torsion theory. The following are equivalent:*

- (1) $\widetilde{F(\mathcal{T})} = \mathcal{T}$;
- (2) $\widetilde{F(\mathcal{F})} = \mathcal{F}$;
- (3) *given $X \in \mathfrak{C}$, X is τ -local if and only if FX is τ -local;*
- (4) $FS_\tau \mathbf{Q}_\tau = S_\tau \mathbf{Q}_\tau F$.

Proof. The equivalence between (1) and (2) follows since $\text{Hom}_{\mathfrak{C}}(A, B) \cong \text{Hom}_{\mathfrak{C}}(F(A), F(B))$, for all $A, B \in \mathfrak{C}$ and by the fact that $\mathcal{F} = \mathcal{T}^\perp$ and $\mathcal{T} = {}^\perp \mathcal{F}$.

(1)&(2) \Rightarrow (3). Given a τ -local $X \in \mathfrak{C}$, one can consider the following exact sequence

$$0 \rightarrow \mathbf{T}_\tau(FX) \rightarrow FX \rightarrow S_\tau \mathbf{Q}_\tau FX \rightarrow T \rightarrow 0,$$

where $T \in \mathcal{T}$. It follows from (2) that $F(X) \in \mathcal{F}$ and so $\mathbf{T}_\tau(FX) = 0$. Furthermore, it is a standard fact that $T \cong \mathbf{T}_\tau(E(FX)/FX)$. Since F is an equivalence, it is exact and it commutes with injective envelopes, so $T \cong \mathbf{T}_\tau(F(E(X)/X))$ which is trivial by the fact that X is τ -local (implying that $E(X)/X \in \mathcal{F}$) and (2). On the other hand, if FX is local, then $\mathbf{T}_\tau(FX) = 0$ and $\mathbf{T}_\tau(F(E(X)/X))$, which is equivalent to say that X is τ -local, by the exactness of F and (1).

(3) \Rightarrow (1). It follows by the fact that the τ -torsion objects are exactly the objects not admitting non-trivial morphisms to a τ -local object.

(1)&(3) \Rightarrow (4). Let $X \in \mathfrak{C}$ and consider the following exact sequence

$$0 \rightarrow \mathbf{T}_\tau(X) \rightarrow X \rightarrow S_\tau \mathbf{Q}_\tau(X) \rightarrow T \rightarrow 0,$$

where $T \in \mathcal{T}$. Applying $\mathbf{Q}_\tau F$ to the above sequence, using the exactness of such functor and (1), one gets $\mathbf{Q}_\tau F(X) \cong \mathbf{Q}_\tau F S_\tau \mathbf{Q}_\tau(X)$. Now, applying S_τ and using (3) we obtain $S_\tau \mathbf{Q}_\tau F(X) \cong S_\tau \mathbf{Q}_\tau F S_\tau \mathbf{Q}_\tau(X) = F S_\tau \mathbf{Q}_\tau(X)$.

(4) \Rightarrow (3). Let $X \in \mathfrak{C}$. Then, X is τ -local if and only if $X \cong S_\tau \mathbf{Q}_\tau(X)$, if and only if $FX \cong F S_\tau \mathbf{Q}_\tau(X)$. Using (4), $FX \cong S_\tau \mathbf{Q}_\tau F(X)$, which is equivalent to say that FX is τ -local. \square

Recall that, for a Gabriel category \mathfrak{C} , the spectrum $\text{Sp}(\mathfrak{C})$ of \mathfrak{C} is defined as the union of all the spectra $\text{Sp}^\alpha(\mathfrak{C}) = \text{Sp}(\mathfrak{C}_{\alpha+1}/\mathfrak{C}_\alpha)$, with $\alpha < \text{G.dim}(\mathfrak{C})$. In particular, $\text{Sp}^{-1}(\mathfrak{C}) = \text{Sp}(\mathfrak{C}_0)$ is a set of representatives of the isomorphism classes of the simple objects in \mathfrak{C} .

In the following lemma we show that the equivalence $F : \mathfrak{C} \rightarrow \mathfrak{C}$ induces a bijection of $\text{Sp}^{-1}(\mathfrak{C})$ onto itself. This fact is then applied in Proposition 3.5 to show that F induces bijections of $\text{Sp}^\alpha(\mathfrak{C})$ onto itself, for all $\alpha < \text{G.dim}(\mathfrak{C})$.

Lemma 3.4. *Let \mathfrak{C} be a Gabriel category. For any simple object S , the object $F(S)$ is again simple. Furthermore, if we define a function*

$$f_{-1} : \text{Sp}^{-1}(\mathfrak{C}) \rightarrow \text{Sp}^{-1}(\mathfrak{C})$$

mapping $S \in \text{Sp}^{-1}(\mathfrak{C})$ to the isomorphism class of $F(S)$ in $\text{Sp}^{-1}(\mathfrak{C})$, then f_{-1} is well-defined and bijective.

Proof. The fact that F sends simples to simples follows by the well-known fact that an equivalence preserves the lattice of subobjects of a given object. For the second part of the statement, just notice that, given two simple objects S_1 and S_2 , $S_1 \cong S_2$ is equivalent to $F(S_1) \cong F(S_2)$. Furthermore, f_{-1} is surjective by the essential surjectivity of F . \square

Proposition 3.5. *Let \mathfrak{C} be a Gabriel category and let $F : \mathfrak{C} \rightarrow \mathfrak{C}$ be a self-equivalence. Then,*

$$(1_\alpha) \quad \widetilde{F(\mathfrak{C}_\alpha)} = \mathfrak{C}_\alpha;$$

(2_α) the functor $F_\alpha : \mathfrak{C}/\mathfrak{C}_\alpha \rightarrow \mathfrak{C}/\mathfrak{C}_\alpha$, defined by the composition $F_\alpha = \mathbf{Q}_\alpha F \mathbf{S}_\alpha$ is an equivalence;

for all $-1 \leq \alpha < \text{G.dim}(\mathfrak{C})$. In particular, via the identification $\text{Sp}^\alpha(\mathfrak{C}) = \text{Sp}(\mathfrak{C}_{\alpha+1}/\mathfrak{C}_\alpha) = \text{Sp}^{-1}(\mathfrak{C}/\mathfrak{C}_\alpha)$, each F_α induces a bijection

$$f_\alpha : \text{Sp}^\alpha(\mathfrak{C}) \rightarrow \text{Sp}^\alpha(\mathfrak{C}),$$

defined as in Lemma 3.4.

Proof. We prove simultaneously (1_α) and (2_α) by transfinite induction on α.

In case $\alpha = -1$, then (1₋₁) just says that $F(\{0\}) = \{0\}$, so it is trivially verified, while (2₋₁) is true as F_{-1} is just F .

Suppose now that (1_α) and (2_α) are verified for some $-1 \leq \alpha < \text{G.dim}(\mathfrak{C})$. Notice that $\mathfrak{C}_{\alpha+1}/\mathfrak{C}_\alpha$ can be identified with $(\mathfrak{C}/\mathfrak{C}_\alpha)_0$, that is, the smallest hereditary torsion subclass of $\mathfrak{C}/\mathfrak{C}_\alpha$ containing all the simple objects of this category. Since F_α is an equivalence, it sends hereditary torsion classes to hereditary torsion classes so the isomorphism closure of $F_\alpha(\mathfrak{C}_{\alpha+1}/\mathfrak{C}_\alpha)$ is exactly $\mathfrak{C}_{\alpha+1}/\mathfrak{C}_\alpha$, as by Lemma 3.4 the image under an equivalence of a class containing all the simples, contains all the simples. Thus, given $X \in \mathfrak{C}$, we have the following equivalences:

$$(X \in \mathfrak{C}_{\alpha+1}) \Leftrightarrow (\bar{X} := \mathbf{Q}_\alpha(X) \in \mathfrak{C}_{\alpha+1}/\mathfrak{C}_\alpha) \Leftrightarrow (\exists \bar{Y} \in \mathfrak{C}_{\alpha+1}/\mathfrak{C}_\alpha : F_\alpha(\bar{Y}) \cong \bar{X}) \Leftrightarrow (\exists Y \in \mathfrak{C}_{\alpha+1} : F_\alpha \mathbf{Q}_\alpha(Y) \cong \bar{X}),$$

where the second equivalence follows by (1_α). Now, to prove (1_{α+1}) we have to show that the last of the above conditions is equivalent to $(\exists Y \in \mathfrak{C}_{\alpha+1} : F(Y) \cong X)$. The implication $(\exists Y \in \mathfrak{C}_{\alpha+1} : F(Y) \cong X) \Rightarrow (\exists Y \in \mathfrak{C}_{\alpha+1} : F_\alpha \mathbf{Q}_\alpha(Y) \cong \bar{X})$ is trivial, as $\mathbf{Q}_\alpha F(Y) \cong \mathbf{Q}_\alpha F \mathbf{S}_\alpha \mathbf{Q}_\alpha(Y) = F_\alpha \mathbf{Q}_\alpha(Y)$. For the converse implication, assume that there exists $Y \in \mathfrak{C}_{\alpha+1}$ such that $F_\alpha \mathbf{Q}_\alpha(Y) \cong \bar{X}$. We have the following diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T_1 & \longrightarrow & F(Y') & \longrightarrow & \mathbf{S}_\alpha \mathbf{Q}_\alpha F(Y') & \longrightarrow & T_2 & \longrightarrow & 0 \\ & & & & \downarrow & & \cong & & & & \\ 0 & \longrightarrow & T_3 & \longrightarrow & X & \longrightarrow & \mathbf{S}_\alpha \mathbf{Q}_\alpha X & \longrightarrow & T_4 & \longrightarrow & 0 \end{array}$$

where $Y' = \mathbf{S}_\alpha \mathbf{Q}_\alpha(Y)$ and $T_1, T_2, T_3, T_4 \in \mathfrak{C}_\alpha$. Using (2_α), one obtains $\mathfrak{C}_\alpha = \widetilde{F(\mathfrak{C}_\alpha)} \subseteq \widetilde{F(\mathfrak{C}_{\alpha+1})}$, so the first line of the diagram says that $\mathbf{S}_\alpha \mathbf{Q}_\alpha F(Y') \in \widetilde{F(\mathfrak{C}_{\alpha+1})}$, the isomorphism then shows that $\mathbf{S}_\alpha \mathbf{Q}_\alpha X \in \widetilde{F(\mathfrak{C}_{\alpha+1})}$. One concludes by the second line that $X \in \widetilde{F(\mathfrak{C}_{\alpha+1})}$, proving (1_{α+1}). In order to show (2_{α+1}) one has to show that $F_{\alpha+1}$ is essentially surjective and fully faithful. The former is verified as follows: take $\bar{X} \in \mathfrak{C}/\mathfrak{C}_{\alpha+1}$, let $X = \mathbf{S}_{\alpha+1} \bar{X}$, choose $Y \in \mathfrak{C}$ such that $F(Y) \cong X$ and let $\bar{Y} = \mathbf{Q}_{\alpha+1} Y$, one concludes that

$$F_{\alpha+1}(\bar{Y}) = \mathbf{Q}_{\alpha+1} F \mathbf{S}_{\alpha+1} \mathbf{Q}_{\alpha+1}(Y) \stackrel{(*)}{\cong} \mathbf{Q}_{\alpha+1} \mathbf{S}_{\alpha+1} \mathbf{Q}_{\alpha+1} F(Y) \cong \mathbf{Q}_{\alpha+1} F(Y) \cong \bar{X},$$

where (*) is given by Lemma 3.3 (4) (using that we already verified (1_{α+1}), which corresponds to part (1) of that lemma). To verify full faithfulness take $X, Y \in \mathfrak{C}/\mathfrak{C}_{\alpha+1}$, then

$$\text{Hom}_{\mathfrak{C}/\mathfrak{C}_{\alpha+1}}(X, Y) \cong \text{Hom}_{\mathfrak{C}}(\mathbf{S}_{\alpha+1} X, \mathbf{S}_{\alpha+1} Y) \cong \text{Hom}_{\mathfrak{C}}(F \mathbf{S}_{\alpha+1} X, F \mathbf{S}_{\alpha+1} Y) \cong \text{Hom}_{\mathfrak{C}/\mathfrak{C}_{\alpha+1}}(F_{\alpha+1} X, F_{\alpha+1} Y),$$

where the last isomorphism follows as, by Lemma 3.3 (3), $F \mathbf{S}_{\alpha+1} X$ and $F \mathbf{S}_{\alpha+1} Y$ are both $(\alpha+1)$ -local.

Finally, let $\lambda \leq \text{G.dim}(\mathfrak{C})$ be a limit ordinal and assume (1_α), (2_α) for all $\alpha < \lambda$. Then, (1_λ) trivially follows recalling that \mathfrak{C}_λ is the smallest hereditary torsion class containing $\mathfrak{C}_\alpha = \widetilde{F(\mathfrak{C}_\alpha)}$ for all $\alpha < \lambda$ and the same description can be given for $\widetilde{F(\mathfrak{C}_\lambda)}$. Furthermore, (2_λ) follows by (1_λ) and Lemma 3.3 exactly as in the successor case. \square

Motivated by the above proposition, we can give the following definition.

Definition 3.6. Given a Gabriel category \mathfrak{C} , a self-equivalence $F : \mathfrak{C} \rightarrow \mathfrak{C}$ and a point $\pi \in \text{Sp}^\alpha(\mathfrak{C}) \subseteq \text{Sp}(\mathfrak{C})$, let

$$\mathcal{O}_F(\pi) = \{f_\alpha^n(\pi) : n \in \mathbb{Z}\}$$

be the F -orbit of π , where $f_\alpha : \text{Sp}^\alpha(\mathfrak{C}) \rightarrow \text{Sp}^\alpha(\mathfrak{C})$ is the bijective map described in Proposition 3.5.

It is clear that each point of the spectrum belongs to a unique F -orbit. In particular, the F -orbits induce a partition of the Gabriel spectrum.

3.2 Characterization of compatible length functions in Gabriel categories

In this subsection we use the orbit decomposition of the Gabriel spectrum and the classification of length functions in Gabriel categories to give a sufficient and necessary condition for a length function to be compatible with a given self-equivalence.

Theorem 3.7. *Let \mathfrak{C} be a Gabriel category, let $F : \mathfrak{C} \rightarrow \mathfrak{C}$ be a self-equivalence and let $L : \mathfrak{C} \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$ be a length function. Then, F and L are compatible if and only if, for all $\pi \in \text{Sp}(\mathfrak{C})$,*

- (1) *if $\pi \in \text{Supp}^\infty(L)$ then $\mathcal{O}_F(\pi) \subseteq \text{Supp}^\infty(L)$;*
- (2) *if $\pi \in \text{Supp}^{fin}(L)$, then*
 - a. $\mathcal{O}_F(\pi) \subseteq \text{Supp}^{fin}(L)$;
 - b. $\lambda(\pi') = \lambda(\pi)$, for all $\pi' \in \mathcal{O}_F(\pi)$.

Let us recall our decomposition of $L : \mathfrak{C} \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$ obtained in Section 2.6. The first thing we did was to define a torsion theory $\tau = (\mathcal{T}, \mathcal{F})$, where $\mathcal{T} = \overline{\text{Fin}}(L)$ is the smallest hereditary torsion class containing $\text{Fin}(L)$. This allowed us to write L a sum of its finite and infinite components:

$$L = L^{fin} + L^\infty,$$

where L^∞ assumes the value 0 on \mathcal{T} and ∞ elsewhere, while L^{fin} is the Vámos extension of the restriction of L to \mathcal{T} .

Similarly we can define a torsion theory $\tau_F = (\mathcal{T}^F, \mathcal{F}^F)$ with $\mathcal{T}^F = \overline{\text{Fin}}(L_F)$ (for the definition of L_F see (3.1)). This induces a decomposition

$$L_F = L_F^{fin} + L_F^\infty.$$

Theorem 3.7 will follow showing that $\tau = \tau_F$ (or, equivalently, $L^\infty = L_F^\infty$) and $L^{fin} = L_F^{fin}$. In the setting of Theorem 3.7, we denote by f_α the self-bijection of $\text{Sp}^\alpha(\mathfrak{C})$ induced by F .

Lemma 3.8. *In the above notation, the following are equivalent:*

- (1) $\tau = \tau_F$, that is, $L^\infty = L_F^\infty$;
- (2) $\pi \in \text{Supp}^\infty(L)$ implies $\mathcal{O}_F(\pi) \subseteq \text{Supp}^\infty(L)$, for all $\pi \in \text{Sp}(\mathfrak{C})$.

Proof. Let $\alpha < \text{G.dim}(\mathfrak{C})$ and choose $\pi \in \text{Sp}^\alpha(\mathfrak{C})$. Then, $\pi \in \text{Supp}_\alpha^\infty(L_F)$ if and only if $L_F^\infty(\mathbf{S}_\alpha(S(\pi))) = \infty$, if and only if $L^\infty(F\mathbf{S}_\alpha(S(\pi))) = \infty$, if and only if $f_\alpha(\pi) \in \text{Supp}_\alpha^\infty(L)$. Thus,

$$\text{Supp}_\alpha^\infty(L_F) = f_\alpha(\text{Supp}_\alpha^\infty(L)).$$

By this equality it is clear that $\text{Supp}_\alpha^\infty(L_F) = \text{Supp}_\alpha^\infty(L)$ if and only if $\text{Supp}_\alpha^\infty(L) = f_\alpha(\text{Supp}_\alpha^\infty(L))$, which is equivalent to affirm that $\text{Supp}_\alpha^\infty(L)$ is f_α and f_α^{-1} -invariant. This happens for all $\alpha < \text{G.dim}(\mathfrak{C})$ if and only if (2) is verified. \square

We can now concentrate on showing that $L^{fin} = L_F^{fin}$ is equivalent to condition (2) in Theorem 3.7:

Lemma 3.9. *In the above notation, the following are equivalent:*

- (1) $L^{fin} = L_F^{fin}$;
- (2) *if $\pi \in \text{Supp}^{fin}(L)$, then*
 - a. $\mathcal{O}_F(\pi) \subseteq \text{Supp}^{fin}(L)$;
 - b. $\lambda(\pi') = \lambda(\pi)$, for all $\pi' \in \mathcal{O}_F(\pi)$.

Proof. For all $\alpha < \text{G.dim}(\mathfrak{C})$, let

$$\overline{L}_\alpha, (\overline{L}_F)_\alpha : \mathfrak{C}_{\alpha+1}/\mathfrak{C}_\alpha \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$$

be the functions described in (2.6) relative to L and L_F respectively. Using Lemma 3.3, one can follow the steps of the construction of L_α in the proof of Lemma 2.34 and show that $(L_F)_\alpha(M) = L_\alpha(F(M))$ for all $M \in \mathfrak{C}_{\alpha+1}$. Thus $(\overline{L}_F)_\alpha(M) = \overline{L}_\alpha(F_\alpha(M))$ for all $M \in \mathfrak{C}_{\alpha+1}/\mathfrak{C}_\alpha$ (where as usual $F_\alpha = Q_\alpha F S_\alpha$). By the structure of length functions in Gabriel categories, $L^{fin} = L_F^{fin}$ if and only if $\overline{L}_\alpha = (\overline{L}_F)_\alpha$, for all $\alpha < \text{G.dim}(\mathfrak{C})$. Choose $\alpha < \text{G.dim}(\mathfrak{C})$ and let $\pi \in \text{Sp}^\alpha(\mathfrak{C})$. Then

$$(\overline{L}_F)_\alpha(S(\pi)) = \overline{L}_\alpha(F_\alpha(S(\pi))) = \overline{L}_\alpha(S(f_\alpha(\pi))).$$

Thus, $\text{Supp}_\alpha^{fin}(L) = f_\alpha(\text{Supp}_\alpha^{fin}(L_F))$ and so $\text{Supp}_\alpha^{fin}(L) = \text{Supp}_\alpha^{fin}(L_F)$ if and only if $\text{Supp}_\alpha^{fin}(L)$ is f_α and f_α^{-1} -invariant, which is condition (2)a. in the statement. Furthermore, given $\pi \in \text{Supp}_\alpha^{fin}(L)$, the constant associated to π in the decomposition of L is $\overline{L}_\alpha(S(\pi))$, while the constant associated to π in the decomposition of L_F is $\overline{L}_\alpha(S(f_\alpha(\pi)))$. Thus the two functions coincide if and only if \overline{L}_α is constant on the simple objects belonging to the same orbit under f_α , that is, condition (2)b in the statement. \square

3.3 Examples

Let K be a division ring and consider the category $K\text{-Mod}$ of left K -modules. $\text{Sp}(K\text{-Mod})$ consist of a single point, thus Theorem 3.7 says that any length function is compatible with any self-equivalence of $K\text{-Mod}$. On the other hand, we already proved in Corollary 2.27 that the length functions on $K\text{-Mod}$ are just multiples of the composition length (which in this case is just the dimension over K) so this is not a very deep result.

The previous example can be generalized as follows. Let \mathfrak{C} be a Gabriel category and consider the composition length $\ell : \mathfrak{C} \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$. Then, $\overline{\text{Fin}}(\ell) = \mathfrak{C}_0$ and so $\text{Supp}^\infty(\ell) = \text{Sp}(\mathfrak{C}/\mathfrak{C}_0) = \bigcup_{\alpha \geq 0} \text{Sp}^\alpha(\mathfrak{C})$, while $\text{Supp}^{fin}(\ell) = \text{Sp}(\mathfrak{C}_0) = \text{Sp}^{-1}(\mathfrak{C})$. Clearly both the finite and the infinite spectrum are invariant under any family of self-bijections $\{f_\alpha : \text{Sp}^\alpha(\mathfrak{C}) \rightarrow \text{Sp}^\alpha(\mathfrak{C}) : \alpha < \text{G.dim}(\mathfrak{C})\}$. Furthermore, the constants associated to each $\pi \in \text{Supp}^{fin}(\ell)$ in the decomposition of ℓ as linear combination of atomic functions are all 1. Thus, Theorem 3.7 can be applied to show that ℓ is compatible with any self-equivalence of \mathfrak{C} .

A further generalization of the above example can be achieved as follows. Let \mathfrak{C} be a Gabriel category, for any $\alpha < \text{G.dim}(\mathfrak{C})$ we define a length function

$$\ell_\alpha : \mathfrak{C} \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\} \quad \text{such that} \quad \ell_\alpha(M) = \ell(\mathbf{Q}_\alpha(M)),$$

where ℓ is the composition length in $\mathfrak{C}/\mathfrak{C}_{\alpha+1}$. One can show that $\text{Supp}^{fin}(\ell_\alpha) = \text{Sp}^\alpha(\mathfrak{C})$ and $\text{Supp}^\infty(\ell_\alpha) = \text{Sp}(\mathfrak{C}/\mathfrak{C}_{\alpha+1}) = \bigcup_{\beta > \alpha} \text{Sp}^\beta(\mathfrak{C})$, furthermore

$$\ell_\alpha = \sum_{\pi \in \text{Sp}^\alpha(\mathfrak{C})} \ell_\pi + \sum_{\pi \in \text{Sp}(\mathfrak{C}/\mathfrak{C}_{\alpha+1})} \infty \cdot \ell_\pi.$$

Theorem 3.7 implies that ℓ_α (and any of its multiples) is compatible with any self-equivalence of \mathfrak{C} .

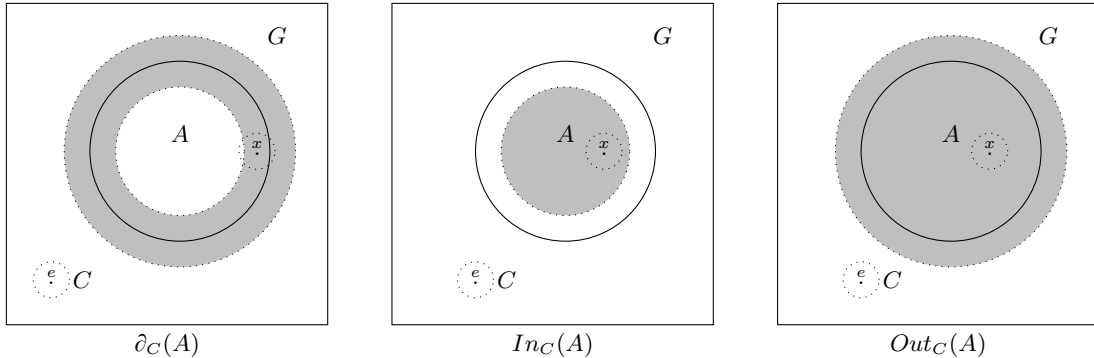
4 Amenable groups and crossed products

4.1 Amenable groups

Amenable groups were defined by John von Neumann in 1929 as groups admitting a left-invariant mean. We adopt here an equivalent definition of amenability given by Følner [13]. Indeed, consider two subsets $A, C \subseteq G$, then

- the C -interior of A is $\text{In}_C(A) = \{x \in G : xC \subseteq A\}$;
- the C -exterior of A is $\text{Out}_C(A) = \{x \in G : xC \cap A \neq \emptyset\}$;
- the C -boundary of A is $\partial_C(A) = \text{Out}_C(A) \setminus \text{In}_C(A)$.

If $e \in C$, one can imagine the above notions as in the following picture



Definition 4.1. A group G is amenable if and only if there exists a directed set (I, \leq) and a net $\{F_i : i \in I\}$ of finite subsets of G such that, for any finite subset C of G ,

$$\lim_I \frac{|\partial_C(F_i)|}{|F_i|} = 0. \quad (4.1)$$

Any such net is called a Følner net.

If G is countable, then one can take $I = \mathbb{N}$ and just speak about *Følner sequences*. Furthermore, given a countable group G and a Følner sequence $\mathfrak{f} = \{F_n : n \in \mathbb{N}\}$, we say that \mathfrak{f} is a *Følner exhaustion* if

- (1) $e \in F_0$ and $F_n \subseteq F_{n+1}$ for all $n \in \mathbb{N}$;
- (2) $\bigcup_{n \in \mathbb{N}} F_n = G$.

It can be proved that a finitely generated group is amenable if and only if it admits a Følner exhaustion (see for example [30, Lemma 5.3]).

Example 4.2. (1) Any finite group and any finitely generated Abelian group is amenable;

(2) consider a finitely generated group G and let $S = S^{-1}$ be a finite set of generators containing the unit of G . For all $n \in \mathbb{N}_+$, let $B_n(S) = \{s_1 \cdot \dots \cdot s_n : s_i \in S\}$ and define a function:

$$f_S : \mathbb{N}_+ \rightarrow \mathbb{N}_+ \text{ such that } f_S(n) = |B_n(S)|.$$

The group G is said to be of sub-exponential growth if the growth of f_S is sub-exponential (it can be shown that this notion does not depend on the choice of the generating set S). If G is of sub-exponential growth, then $\{B_n(S) : n \in \mathbb{N}_+\}$ is a Følner exhaustion for G (see for example [4, Section 6.11]);

- (3) Non-commutative free groups are not amenable. So, any group which contains a free subgroup of rank 2 is not amenable. There exist amenable groups with exponential growth.
- (4) It is known that the class of amenable groups is closed under the operations of taking subgroups, taking factors over normal subgroups, taking extensions and taking increasing unions. We obtain that a group G is amenable if and only if all its finitely generated subgroups are amenable, in particular, arbitrary Abelian groups and locally finite groups are amenable.

4.2 Non-negative real functions on finite subsets of an amenable group

In their seminal paper [27], Ornstein and Weiss introduced a notion of entropy for actions of amenable groups on metric spaces. Using the theory of quasi-tiles, they were able to prove that the lim sup defining their entropy is a true limit. In what follows, we recall some of these deep results as they can be applied with just minor changes to our algebraic setting. The following terminology and results are due to Ornstein and Weiss [27] (see also [19] and [37]).

Denote by $\mathcal{F}(G)$ the family of all finite subsets of G . Let $A_1, \dots, A_k \in \mathcal{F}(G)$ and $\varepsilon \in (0, 1)$. The family $\{A_1, \dots, A_k\}$ is ε -disjoint if there are $A'_1, \dots, A'_k \in \mathcal{F}(G)$ such that

- (1) $A'_i \subseteq A_i$ and $|A'_i|/|A_i| > 1 - \varepsilon$ for $i = 1, \dots, k$;
- (2) $A'_i \cap A'_j = \emptyset$ if $1 \leq i \neq j \leq k$.

Given $\alpha \in (0, 1]$ and $A \in \mathcal{F}(G)$, $\{A_1, \dots, A_k\}$ is an α -cover of A if

$$\frac{|A \cap (\bigcup_{i=1}^k A_i)|}{|A|} \geq \alpha.$$

Finally, $\{A_1, \dots, A_k\}$ ε -quasi-tiles A if there exist $C_1, \dots, C_k \in \mathcal{F}(G)$ such that

- (1) $C_i A_i \subseteq A$, for all $i = 1, \dots, k$, and $\{c A_i : c \in C_i\}$ forms an ε -disjoint family;
- (2) $C_i A_i \cap C_j A_j = \emptyset$, if $1 \leq i \neq j \leq k$;
- (3) $\{C_i A_i : i = 1, \dots, k\}$ forms a $(1 - \varepsilon)$ -cover of A .

The subsets C_1, \dots, C_k are called *tiling centers*. It is a deep result, due to Ornstein and Weiss, that whenever G is an amenable group and $\{F_n\}_{n \in \mathbb{N}}$ is a Følner exhaustion, for any (small enough) $\varepsilon > 0$, one can find a nice family of subsets of G that ε -quasi-tiles F_n for all (big enough) $n \in \mathbb{N}$. More precisely:

Theorem 4.3. [27] *Let G be a finitely generated amenable group and let $\{F_n\}_{n \in \mathbb{N}}$ be a Følner exhaustion of G . Then, for all $\varepsilon \in (0, 1/4)$ and $\bar{n} \in \mathbb{N}$, there exist positive integers n_1, \dots, n_k such that $\bar{n} \leq n_1 \leq \dots \leq n_k$ and $\{F_{n_1}, \dots, F_{n_k}\}$ ε -quasi-tiles F_n , for all $n \geq \bar{n}$.*

In the rest of the subsection we recall some results and terminology about non-negative invariants for the finite subsets of G , that is, functions $f : \mathcal{F}(G) \rightarrow \mathbb{R}_{\geq 0}$. In particular, we say that f is

- (1) *monotone* if $f(A) \leq f(A')$, for all $A \subseteq A' \in \mathcal{F}(G)$;
- (2) *sub-additive* if $f(A \cup A') \leq f(A) + f(A')$, for all $A, A' \in \mathcal{F}(G)$;
- (3) *(left) G -equivariant* if $f(gA) = f(A)$, for all $A \in \mathcal{F}(G)$ and $g \in G$.

A consequence of (3) above is that $f(\{g\}) = f(\{e\})$, for all $g \in G$. Thus by (2), $f(A) \leq \sum_{g \in A} f(\{g\}) = |A|f(e)$, for all $A \in \mathcal{F}(G)$.

The following result, generally known as “Ornstein-Weiss Lemma”, is proved in [27] for a suitable class of locally compact amenable groups (a direct proof, along the same lines, in the discrete case can be found in [37], while a nice alternative argument, based on ideas of Gromov, is given in [22]).

Lemma 4.4. *Let G be a finitely generated amenable group and consider a monotone, sub-additive and G -equivariant function $f : \mathcal{F}(G) \rightarrow \mathbb{R}_{\geq 0}$. Then, for any Følner sequence $\{F_n\}_{n \in \mathbb{N}}$, the sequence $(f(F_n)/|F_n|)_{n \in \mathbb{N}}$ converges and the value of the limit $\lim_{n \in \mathbb{N}} f(F_n)/|F_n|$ is the same for any choice of the Følner sequence.*

We conclude this paragraph with the following consequence of Theorem 4.3. The proof is essentially given in [19], we give it here for completeness sake, as our statement is slightly different.

Corollary 4.5. *Let G be a finitely generated amenable group and $\{F_n\}_{n \in \mathbb{N}}$ be a Følner exhaustion of G . Then, for any $\varepsilon \in (0, 1/4)$ and $\bar{n} \in \mathbb{N}$ there exist integers n_1, \dots, n_k such that $\bar{n} \leq n_1 \leq \dots \leq n_k$ and, for any sub-additive and G -equivariant $f : \mathcal{F}(G) \rightarrow \mathbb{R}$ we have*

$$\limsup_{n \rightarrow \infty} \frac{f(F_n)}{|F_n|} \leq M\varepsilon + \frac{1}{1-\varepsilon} \cdot \max_{1 \leq i \leq k} \frac{f(F_{n_i})}{|F_{n_i}|},$$

where $M = f(\{e\})$.

It is important to underline that the choice of the n_1, \dots, n_k does not depend on the function f , but we can really find a family $\{n_1, \dots, n_k\}$, which works for all f at the same time.

Proof. Let $\varepsilon \in (0, 1/4)$ and $\bar{n} \in \mathbb{N}$. By Theorem 4.3, there exist positive integers n_1, \dots, n_k such that $\bar{n} \leq n_1 \leq \dots \leq n_k$ and $\{F_{n_1}, \dots, F_{n_k}\}$ ε -quasi-tiles F_n , for all $n \geq \bar{n}$. We let C_1^1, \dots, C_k^n be the tiling centers for F_n . Thus, when $n \geq \bar{n}$, we have

$$F_n \supseteq \bigcup_{i=1}^k C_i^n F_{n_i} \quad \text{and} \quad \left| \bigcup_{i=1}^k C_i^n F_{n_i} \right| \geq \max \left\{ (1-\varepsilon)|F_n|, (1-\varepsilon) \sum_{i=1}^k |C_i^n| \cdot |F_{n_i}| \right\}.$$

Now, let $f : \mathcal{F}(G) \rightarrow \mathbb{R}_{\geq 0}$ be a sub-additive and G -equivariant function, we obtain that

$$\begin{aligned} \frac{f(F_n)}{|F_n|} &\leq \frac{f(F_n \setminus \bigcup_{i=1}^k C_i^n F_{n_i})}{|F_n|} + \frac{f(\bigcup_{i=1}^k C_i^n F_{n_i})}{|F_n|} \leq M \cdot \frac{|F_n \setminus \bigcup_{i=1}^k C_i^n F_{n_i}|}{|F_n|} + \frac{f(\bigcup_{i=1}^k C_i^n F_{n_i})}{|\bigcup_{i=1}^k C_i^n F_{n_i}|} \\ &\leq M\varepsilon + \frac{f(\bigcup_{i=1}^k C_i^n F_{n_i})}{|\bigcup_{i=1}^k C_i^n F_{n_i}|} \leq M\varepsilon + \frac{\sum_{i=1}^k |C_i^n| f(F_{n_i})}{(1-\varepsilon) \sum_{i=1}^k |C_i^n| \cdot |F_{n_i}|} \leq M\varepsilon + \frac{1}{1-\varepsilon} \cdot \max_{1 \leq i \leq k} \frac{f(F_{n_i})}{|F_{n_i}|}, \end{aligned}$$

as desired. \square

4.3 Crossed products

Given a group G and a ring R , a *crossed product* $R * G$ of R with G is a ring constructed as follows: as a set, $R * G$ is the collection of all the formal sums of the form

$$\sum_{g \in G} r_g \underline{g},$$

with $r_g \in R$ and $r_g = 0$ for all but finite $g \in G$, and where the \underline{g} are symbols uniquely assigned to each $g \in G$. Sum in $R * G$ is as expected, that is, it is defined component-wise exploiting the addition in R :

$$\left(\sum_{g \in G} r_g \underline{g} \right) + \left(\sum_{g \in G} s_g \underline{g} \right) = \sum_{g \in G} (r_g + s_g) \underline{g}.$$

In order to define a product in $R * G$, we need to take two maps

$$\sigma : G \rightarrow \text{Aut}_{\text{ring}}(R) \quad \text{and} \quad \rho : G \times G \rightarrow U(R),$$

where $\text{Aut}_{\text{ring}}(R)$ denotes the group of automorphisms of R in the category of (unitary) rings, while $U(R)$ is the group of units of R . Given $g \in G$ and $r \in R$ we denote the image of r via the automorphism $\sigma(g)$ by $r^{\sigma(g)}$. We suppose also that σ and ρ satisfy the following conditions for all $r \in R$ and g_1, g_2 and $g_3 \in G$:

- (Cross.1) $\rho(g_1, g_2) \rho(g_1 g_2, g_3) = \rho(g_2, g_3)^{\sigma(g_1)} \rho(g_1, g_2 g_3)$;
- (Cross.2) $r^{\sigma(g_2) \sigma(g_1)} = \rho(g_1, g_2) r^{\sigma(g_1 g_2)} \rho(g_1, g_2)^{-1}$;
- (Cross.3) $\rho(g, e) = \rho(e, g) = 1$ (for all $g \in G$) and $\sigma(e) = 1$.

The product in $R * G$ is defined by the rule $(r \underline{g})(s \underline{h}) = r s^{\sigma(g)} \rho(g, h) \underline{gh}$, together with bilinearity, that is

$$\left(\sum_{g \in G} r_g \underline{g} \right) \left(\sum_{g \in G} s_g \underline{g} \right) = \sum_{g \in G} \left(\sum_{h_1 h_2 = g} r_{h_1} s_{h_2}^{\sigma(h_1)} \rho(h_1, h_2) \right) \underline{g}.$$

By (Cross.1) and (Cross.2) above, $R * G$ is an associative and unitary ring, while by (Cross.3) $1_{R * G} = \underline{e}$.

Of course the definition of $R * G$ does not depend only on R and G , as the choice of σ and ρ is fundamental for defining the product. Anyway one avoids a notation like $R[G, \rho, \sigma]$ and uses the more compact (though imprecise) $R * G$. Of course, the easiest example of crossed group ring is the group ring $R[G]$, which corresponds to trivial maps σ and ρ . For more details on this kind of construction we refer to [29].

Notice that there is a canonical injective ring homomorphism

$$R \rightarrow R * G \quad r \mapsto r \underline{e}.$$

In view of this embedding we identify R with a subring of $R * G$. This allows one to consider a forgetful functor from $R * G\text{-Mod} \rightarrow R\text{-Mod}$.

On the other hand, in general there is no natural map $G \rightarrow R * G$ which is compatible with the operations, anyway the obvious assignment $g \mapsto \underline{g}$ respects the operations *modulo some units of R* . Similarly, given a left $R * G$ -module M , there is a canonical map

$$\lambda : G \rightarrow \text{Aut}_{\mathbb{Z}}(M) \quad , \quad g \mapsto \lambda_g \quad , \quad \lambda_g(m) = \underline{g}m. \quad (4.2)$$

which is not in general a homomorphism of groups. Given an R -submodule ${}_R K \leq M$ and an element $g \in G$, $\lambda_g(K)$ is again an R -submodule of M but it is not in general isomorphic to ${}_R K$. As described in (3.2), there is a self-equivalence of the category $R\text{-Mod}$, induced by the ring automorphism $\sigma(g)$

$$F_{\sigma(g)} : R\text{-Mod} \rightarrow R\text{-Mod}.$$

It follows by the definitions that $\lambda_g(K) \cong \underline{g}K$. In particular, if L is a length function compatible with the equivalence $F_{\sigma(g)}$, then $L(\lambda_g(K)) = L(\underline{g}K)$. This useful fact motivates the following

Definition 4.6. *Let $R * G$ be a given crossed product and let $L : R\text{-Mod} \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$ be a length function. L is said to be compatible with $R * G$ provided L is compatible with $F_{\sigma(g)}$, for all $g \in G$.*

4.4 The action of G on monoids of submodules

Let R be a ring, let G a group, fix a crossed product $R * G$ and let M be a left $R * G$ -module. The set $\mathcal{L}(M)$ of R -submodules of M is a bounded modular and complete lattice (with order induced by the inclusion of submodules). In fact, this lattice structure is more than what we really need: we just need to know that the least upper bound of two elements $N, N' \in \mathcal{L}(M)$ is given by $N + N'$. This operation makes $\mathcal{L}(M) = (\mathcal{L}(M), +, 0)$ into a commutative monoid.

In general, given a category \mathfrak{C} , an object $M \in \mathfrak{C}$ and a group G , a *left action* of G on M is a group homomorphism from G to $\text{Aut}_{\mathfrak{C}}(M)$. In particular, the fact that the map $\lambda : G \rightarrow \text{Aut}_{\mathbb{Z}}(M)$ given by the $R * G$ -module structure on M (see (4.2)) is not a homomorphism of groups is telling us that the group G is not really acting on M . On the other hand, the structure of $R * G$ -module induces an action of G on the monoid $\mathcal{L}(M)$:

$$\lambda : G \rightarrow \text{Aut}(\mathcal{L}(M)) \quad , \quad \lambda(g) = (\lambda_g : \mathcal{L}(M) \rightarrow \mathcal{L}(M)) \quad , \quad (4.3)$$

where $\text{Aut}(\mathcal{L}(M))$ is the group of monoid automorphisms of $\mathcal{L}(M)$ and $\lambda_g(K) = \{\underline{g}k : k \in K\}$, for all $g \in G$ and $K \in \mathcal{L}(M)$.

Lemma 4.7. *In the above notation, $\lambda_g : \mathcal{L}(M) \rightarrow \mathcal{L}(M)$ is a monoid automorphism for all $g \in G$ and $\lambda : G \rightarrow \text{Aut}(\mathcal{L}(M))$ is a group homomorphism.*

Proof. Given $g \in G$, it is easily seen that $\lambda_g(K_1 + K_2) = \lambda_g(K_1) + \lambda_g(K_2)$, for all $K_1, K_2 \in \mathcal{L}(M)$, while $\lambda_{g^{-1}}$ is the inverse of λ_g , so λ_g is a monoid automorphism. Now, let $g, h \in G$ and let $K \in \mathcal{L}(M)$, then

$$\lambda_g \lambda_h(K) = \underline{g} \underline{h} K = \rho(g, h) \underline{g} \underline{h} K = \underline{g} \underline{h} \rho(g, h)^{\sigma(g h)} K = \underline{g} \underline{h} K = \lambda_{gh}(K) \quad ,$$

since $\rho(g, h)^{\sigma(g h)}$ is a unit of R , showing that λ is a group homomorphism. \square

For any subset $F \subseteq G$ and any $K \in \mathcal{L}(M)$, the F -th λ -trajectory of K is

$$T_F(\lambda, K) = \sum_{g \in F} \lambda_g(K) \quad . \quad (4.4)$$

Notice that $T_F(\lambda, K) \in \mathcal{L}(M)$. The (full) λ -trajectory of K is $T_G(\lambda, K)$.

Lemma 4.8. *In the above notation, ${}_{R * G} M$ is finitely generated as a left $R * G$ -module if and only if there exists a finitely generate R -submodule $K \in \mathcal{L}(M)$ such that $M = T_G(\lambda, K)$.*

Proof. If ${}_{R * G} M$ is finitely generated as a left $R * G$ -module then choose a finite set of generators x_1, \dots, x_n , so that, $M = R * G x_1 + \dots + R * G x_n = T_G(\lambda, x_1 R + \dots + x_n R)$. On the other hand, if $M = T_G(\lambda, K)$ with K finitely generated, then any finite set of generators of K generates M as $R * G$ -module. \square

Consider now an $R * G$ -submodule ${}_{R * G} N \leq M$. Any R -submodule of N is also an R -submodule of M , this allows us to identify $\mathcal{L}(N)$ with a sub-monoid of $\mathcal{L}(M)$. One has the following left actions given by the $R * G$ -module structures

$$\lambda' : G \rightarrow \text{Aut}(\mathcal{L}(N)) \quad , \quad \lambda : G \rightarrow \text{Aut}(\mathcal{L}(M)) \quad \text{and} \quad \bar{\lambda} : G \rightarrow \text{Aut}(\mathcal{L}(M/N)) \quad .$$

The following easy lemma will be useful in many situations.

Lemma 4.9. *In the above notation, let $F \leq G$ be a subset, then there is a short exact sequence of left R -modules*

$$0 \rightarrow T_F(\lambda, K) \cap N \rightarrow T_F(\lambda, K) \rightarrow T_F(\bar{\lambda}, (K + N)/N) \rightarrow 0 \quad . \quad (4.5)$$

Proof. The non-trivial maps in (4.5) are induced by the embedding $N \rightarrow M$ and by the projection $M \rightarrow M/N$. One can verify that the resulting sequence is exact noticing that $T_F(\bar{\lambda}, (K + N)/N) = (T_F(\lambda, K) + N)/N$, in fact, $\underline{g}_1(k_1 + N) + \dots + \underline{g}_n(k_n + N) = (\underline{g}_1 k_1 + \dots + \underline{g}_n k_n) + N$ for all $k \in \mathbb{N}_+$, $k_1, \dots, k_n \in K$ and $g_1, \dots, g_n \in F$. \square

Let now $L : R\text{-Mod} \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$ be a length function compatible with $R * G$. We let

$$\text{Fin}_L(M) = \{{}_R K \leq M : L(K) < \infty\} \subseteq \mathcal{L}(M) \quad .$$

By the additivity of L , $\text{Fin}_L(M)$ is a sub-monoid of $\mathcal{L}(M)$. Furthermore, by the compatibility of L with $R * G$, $L(\lambda_g(K)) = L(K)$, thus $\lambda_g(K) \in \text{Fin}_L(M)$. So, there is an induced action

$$\lambda : G \rightarrow \text{Aut}(\text{Fin}_L(M)) \quad , \quad \lambda(g) = \lambda_g : \text{Fin}_L(M) \rightarrow \text{Fin}_L(M) \quad .$$

The length function L is said to be *discrete* provided the set of finite values of L is order-isomorphic to \mathbb{N} . In case L is discrete, the monoid $\text{Fin}_L(M)$ carries redundant information for our needs, for this reason it is usually useful to reduce to a smaller monoid consisting of the finitely generated modules in $\text{Fin}_L(M)$. The following lemma, which is an immediate consequence of the upper continuity and of the discreteness of L , allows for such reduction.

Lemma 4.10. *In the above notation, suppose that L is discrete and that $K \in \text{Fin}_L(M)$. Then, there exists a finitely generated $K' \in \text{Fin}_L(M)$ such that $L(K') = L(K)$.*

We need to introduce a last tool on the monoid of submodules $\mathcal{L}(M)$, that is, a closure operator. Indeed, we consider the torsion class $\text{Ker}(L)$ of all left R -modules K such that $L(K) = 0$. The torsion functor relative to this class was denoted in [32] by $z_L : R\text{-Mod} \rightarrow \text{Ker}(L)$, where, given $K \in R\text{-Mod}$,

$$z_L(K) = \{x \in K \mid L(Rx) = 0\};$$

$z_L(K)$ is called the L -singular submodule of K . If $z_L(K) = K$ (or, equivalently, $L(K) = 0$) we say that M is L -singular. There is a standard technique to associate a closure operator to any given torsion class (see [34]). In particular, given $K \in \mathcal{L}(M)$, we let $\pi : M \rightarrow M/K$ be the natural projection and we define

$$K_{L*} = \pi^{-1}(z_L(M/K))$$

to be the L -purification of K in M . An element $K \in \mathcal{L}(M)$ is said to be L -pure if $K_{L*} = K$, while, if $K \leq K' \in \mathcal{L}(M)$, we say that N is L -essential in K' if $L(K'/K) = 0$, that is, if $K \leq K' \leq K_{L*}$. With this terminology we can reformulate Lemma 4.10 as follows:

Corollary 4.11. *In the above notation, suppose that L is discrete and that $K \in \text{Fin}_L(M)$. Then, K has an L -essential finitely generated submodule.*

We collect in the following lemma some useful properties of L -purifications, which follow by the fact that $(-)_L$ is a closure operator associated to a hereditary torsion theory (see for example [34]).

Lemma 4.12. *Let $M \in R\text{-Mod}$ and consider two submodules K_1 and $K_2 \in \mathcal{L}(M)$. Then,*

- (1) $((K_1)_{L*})_{L*} = (K_1)_{L*}$;
- (2) $\left(\frac{K_1 + K_2}{K_1}\right)_{L*} = \frac{(K_1 + K_2)_{L*}}{K_1}$;
- (3) $(K_1)_{L*} + (K_2)_{L*} \subseteq (K_1 + K_2)_{L*}$ and $((K_1)_{L*} + K_2)_{L*} = (K_1 + K_2)_{L*}$;
- (4) $L((K_1 + K_2)/(K'_1 + K'_2)) = 0$ whenever $K'_1 \leq K_1$, $K'_2 \leq K_2$ are L -essential submodules.

5 The algebraic L -entropy

All along this section, we fix a ring R , an infinite finitely generated amenable group G and a crossed product $R * G$. Furthermore, we fix a length function $L : R\text{-Mod} \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$ with the following two properties:

- (1) it is *discrete*, that is, the image of L is a subset of $\mathbb{R}_{\geq 0} \cup \{\infty\}$ order-isomorphic to $\mathbb{N} \cup \{\infty\}$;
- (2) it is compatible with $R * G$ (see Definition 4.6).

We remark that, if R is a (skew) field, then $L = \dim_R$ satisfies the above conditions for any choice of the functions defining $R * G$; more generally, this happens for all the functions ℓ_α described in Subsection 3.3. On the other hand, if $R * G = R[G]$, then condition (2) is trivially satisfied by any length function L .

5.1 Definition and existence

In this subsection we define the algebraic entropy as an invariant for left $R * G$ -modules. This invariant is modeled to encode the dynamics of the L -finite R -submodules of an $R * G$ -module M , under the action $\lambda : G \rightarrow \text{Aut}(\text{Fin}_L(M))$. It turns out that this notion is not well-behaved on all the $R * G$ -modules but just on a subclass of $R * G\text{-Mod}$ of all the $R * G$ -modules M for which $\text{Fin}_L(M)$ is big enough. More precisely,

Definition 5.1. *A left R -module M is said to be locally L -finite if $\text{Fin}_L(M)$ contains all the finitely generated submodules of M . We denote by $\text{lFin}(L)$ the class of all the locally L -finite left R -modules while we let $\text{lFin}_L(R * G)$ be the class of all the left $R * G$ -modules ${}_R M$ such that ${}_R M \in \text{lFin}(L)$, that is*

$$\text{lFin}_L(R * G) = \{M \in R * G\text{-Mod} : L(K) < \infty \text{ for any f.g. } {}_R K \leq M\}.$$

Notice that $\text{lFin}(L)$ is closed under taking direct limits, quotients and submodules but not in general under taking extensions (see [32]).

Let $M \in \text{lFin}_L(R * G)$ and, as usual, consider the action of G on the L -finite submodules

$$\lambda : G \rightarrow \text{Aut}(\text{Fin}_L(M)).$$

Given $K \in \text{Fin}_L(M)$ and $F \in \mathcal{F}(G)$, one can use the additivity of L to show that $T_F(\lambda, K) \in \text{Fin}_L(M)$ (see (4.4)). In particular, for any L -finite submodule K of M we have a function

$$f_K : \mathcal{F}(G) \rightarrow \mathbb{R}_{\geq 0} \quad f_K(F) = L(T_F(\lambda, K)).$$

We now verify that the above function satisfies the hypotheses of the Ornstein-Weiss Lemma.

Lemma 5.2. *In the above notation, f_K is monotone, sub-additive and G -equivariant.*

Proof. Let $F, F' \in \mathcal{F}(G)$. The proof follows by the properties of L , in fact, $T_F(\lambda, K) + T_{F'}(\lambda, K) = T_{F \cup F'}(\lambda, K)$ and so $f_K(F) + f_K(F') \geq f_K(F \cup F')$, proving sub-additivity. Furthermore, if $F \subseteq F'$ then $T_F(\lambda, K) \leq T_{F'}(\lambda, K)$ and so $f_K(F) \leq f_K(F')$, proving that f_K is monotone. Finally, since L is compatible with $R * G$, for any given $g \in G$ we have $L(T_{gF}(\lambda, K)) = L(\lambda_g(T_F(\lambda, K))) = L(T_F(\lambda, K))$, proving that f_K is G -equivariant. \square

Thus, by the Ornstein-Weiss Lemma, the limit in the following definition exists and it does not depend on the choice of the Følner sequence.

Definition 5.3. *Let $M \in \text{lFin}_L(R * G)$, let $\lambda : G \rightarrow \text{Aut}(\text{Fin}_L({}_R M))$ be the group homomorphism induced by the $R * G$ -module structure on M , let $\{F_n\}_{n \in \mathbb{N}}$ be a Følner sequence for G and let $K \in \text{Fin}_L({}_R M)$. The L -entropy of ${}_R * G M$ with respect to K is*

$$\text{ent}_L(\lambda, K) = \lim_{n \rightarrow \infty} \frac{L(T_{F_n}(\lambda, K))}{|F_n|}.$$

The (algebraic) L -entropy of the $R * G$ -module ${}_R * G M$ is $\text{ent}_L({}_R * G M) = \sup\{\text{ent}_L(\lambda, K) : K \in \text{Fin}_L(M)\}$.

Remark 5.4. We defined the algebraic L -entropy for left $R * G$ -modules in case G is finitely generated. Anyway, the exact same procedure allows one to define this invariant when G is just countable (but not necessarily finitely generated). Furthermore, standard variations of the above arguments using Følner nets allow one to define a similar invariant in case G is not countable.

5.2 Basic properties

In this subsection we study the basic properties of the algebraic L -entropy. For simplicity we fix all along this subsection a locally L -finite left $R * G$ -module ${}_R * G M$. We also denote by $\lambda : G \rightarrow \text{Aut}(\text{Fin}_L(M))$ the action of G induced by the $R * G$ -module structure on M .

Example 5.5. If $L({}_R M) < \infty$, then $\text{ent}_L({}_R * G M) = 0$. In fact, if ${}_R K \leq M$ is any L -finite R -submodule of M , then by definition $\text{ent}_L(\lambda, K) \leq \lim_{n \rightarrow \infty} L(M)/|F_n| \leq \lim_{n \rightarrow \infty} L(M)/n = 0$ (for the second inequality use the fact that, as G is infinite we can take a Følner sequence such that $F_n \not\subseteq F_{n+1}$ for all $n \in \mathbb{N}$, thus $|F_n| \geq n$).

The following result allows us to redefine the algebraic entropy in terms of finitely generated submodules.

Proposition 5.6. *Let $K \in \text{Fin}_L(M)$ and $H \leq K$ be an L -essential submodule. Then*

- (1) $\text{ent}_L(\lambda, H) = \text{ent}_L(\lambda, K)$;
- (2) $\text{ent}_L({}_R * G M) = \sup\{\text{ent}_L(\lambda, K) : {}_R K \text{ finitely generated}\}$.

Proof. By definition of L -essential submodule we have that $K/H \in \text{Ker}(L)$. Furthermore, for all $g \in G$, $\lambda_g K / \lambda_g H \cong F_{\sigma(g)}(K/H)$ so, as by hypothesis L is compatible with $F_{\sigma(g)}$, also $\lambda_g K / \lambda_g H \in \text{Ker}(L)$. In particular, $\lambda_g H$ is L -essential in $\lambda_g K$ for all $g \in G$. By Lemma 4.12(4) and the additivity of L ,

$$L(T_{F_n}(\lambda, K)) = L(T_{F_n}(\lambda, H)),$$

for all $n \in \mathbb{N}$, where $\{F_n\}_{n \in \mathbb{N}}$ is a Følner sequence. Therefore, $\text{ent}_L(\lambda, K) = \text{ent}_L(\lambda, H)$. We can now verify part (2). Indeed, the “ \leq ” inequality comes directly from the definition of entropy. On the other hand, by Lemma 4.10 any L -finite submodule K of M has an L -essential finitely generated submodule H and by part (1) $\text{ent}_L(\lambda, K) = \text{ent}_L(\lambda, H)$, which easily yields the claim. \square

The definition of entropy in terms of finitely generated submodules given in Proposition 5.6(2) allows us to prove many important properties. In the following lemma we show that the entropy is monotone under taking submodules and quotients.

Lemma 5.7. *Let $N \leq M$ be an $R*G$ -submodule. Then*

- (1) $\text{ent}_L(R*G M) \geq \text{ent}_L(R*G N)$;
- (2) $\text{ent}_L(R*G M) \geq \text{ent}_L(R*G(M/N))$.

Proof. We denote by $\lambda' : G \rightarrow \text{Aut}(\text{Fin}_L(N))$ and $\bar{\lambda} : G \rightarrow \text{Aut}(\text{Fin}_L(M/N))$ the actions induced by λ . To prove (1) it is enough to notice that, whenever $K \leq N$ is an L -finite submodule of N , then it is also an L -finite submodule of M and $\text{ent}_L(\lambda', K) = \text{ent}_L(\lambda, K) \leq \text{ent}_L(R*G M)$. Let us verify part (2), using the definition of entropy in terms of finitely generated submodules. Indeed, given a finitely generated submodule $\bar{K} \leq M/N$, there exists a finitely generated (thus L -finite) submodule $K \leq M$ such that $(K + N)/N \cong \bar{K}$. Given a Følner sequence $\{F_n\}_{n \in \mathbb{N}}$, by Lemma 4.9,

$$T_{F_n}(\bar{\lambda}, \bar{K}) = (T_{F_n}(\lambda, K) + N)/N \quad \text{and so} \quad L(T_{F_n}(\bar{\lambda}, \bar{K})) \leq L(T_{F_n}(\lambda, K)) \quad \text{for all } n \in \mathbb{N}.$$

Dividing by $|F_n|$ and taking the limit with $n \rightarrow \infty$ we get $\text{ent}_L(\bar{\lambda}, \bar{K}) \leq \text{ent}_L(\lambda, K) \leq \text{ent}_L(R*G M)$. \square

The above lemma has a converse in some particular situation:

Lemma 5.8. *Let $N \leq M$ be an $R*G$ -submodule. Then,*

- (1) $\text{ent}_L(R*G M) = \text{ent}_L(R*G N)$, *provided* $L(M/N) = 0$;
- (2) $\text{ent}_L(R*G M) = \text{ent}_L(R*G(M/N))$, *provided* $L(N) = 0$.

Proof. One inequality of both statements follows by Lemma 5.7, thus we have just to verify the other one.

Part (1) follows by Proposition 5.6, noticing that, whenever $K \leq M$ is an L -finite submodule, then $K \cap N$ is L -essential in K , thus, $\text{ent}_L(\lambda, K) = \text{ent}_L(\lambda, K \cap N) = \text{ent}_L(\lambda', K \cap N) \leq \text{ent}_L(R*G N)$ (where $\lambda' : G \rightarrow \text{Aut}(\text{Fin}_L(N))$ is the obvious action).

Denote by $\bar{\lambda} : G \rightarrow \text{Aut}(\text{Fin}_L(M/N))$ the obvious action. Let $K \in \text{Fin}_L(M)$ and let $\{F_n\}_{n \in \mathbb{N}}$ be a Følner sequence. The short exact sequence in Lemma 4.9 shows that $L(T_{F_n}(\lambda, K)) = L(T_{F_n}(\bar{\lambda}, \bar{K})) + (T_{F_n}(\lambda, K) \cap N) = L(T_{F_n}(\bar{\lambda}, \bar{K}))$. The conclusion follows. \square

5.3 Historical notes on algebraic entropy

In this subsection we present some of the crucial steps of the development of the notion of algebraic entropy to make it clear where some of the ideas of the present paper come from.

In 1965 Adler, Konheim and McAndrew [1] introduced the topological entropy of a continuous self-map of a compact space. In the final part of their paper, Adler et al. suggested a notion of entropy for an endomorphism of a discrete torsion Abelian group, which is usually denoted by ent . In 1974, Weiss [38] studied some of the basic properties of ent , while in 2009 Dikranjan, Goldsmith, Salce and Zanardo [9] deeply studied this invariant proving also an Addition Theorem and an axiomatic characterization for ent .

The main disadvantage of ent is that of being trivial on endomorphisms of torsion-free discrete Abelian groups. This fact comes directly from the definition, which is based on the cardinality of finite subgroups, that are all trivial in a torsion-free group. In the literature there are two main strategies to remedy this fact and find a suitable additive invariant for the endomorphisms of torsion-free Abelian groups. The first one, proposed by Peters [31] in 1979, consists in substituting the finite subgroups in the definition of ent by finite subsets. The invariant arising this way is very useful in many situations but it seems difficult to generalize this kind of approach to modules over general rings (we refer to [8] and [15] for direct proofs of all the main properties of such invariant). The second strategy was proposed in 2009 by Salce and Zanardo [33]; they suggested to substitute the invariant $\log |\cdot| : \mathbb{Z}\text{-Mod} \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$ by the torsion-free rank of Abelian groups, obtaining the additive invariant ent_{rk} of endomorphisms of Abelian groups. Furthermore, Salce and Zanardo defined a notion of entropy based on some invariants of categories of modules with hypotheses weaker than that of being a length function. In 2013, Salce, Vámos and the author [32], had the intuition that the notion of length function was the missing piece of the puzzle. Using the notion of length function we were able to prove the first general Addition Theorem for the L -entropy of endomorphisms of left R -modules, with R any ring and L a discrete length function of $R\text{-Mod}$.

All the notions of entropy described above are invariants of endomorphisms of modules over some ring R , that is, invariants of $R[X]\text{-Mod}$. Entropies for multiple morphisms acting on a module also have a long history. On the topological side, the foundations of this theory can be found in the seminal paper [27] of Ornstein and Weiss. On the other hand, to the best of the author's knowledge, the present paper is the first attempt to develop a theory of algebraic entropy for group actions on modules.

6 The algebraic entropy is a length function

The aim of this section is to prove the following

Theorem 6.1. *Let R be a ring, let G be a finitely generated infinite amenable group, fix a crossed product $R*G$ and let $L : R\text{-Mod} \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$ be a discrete length function compatible with $R*G$. The invariant $\text{ent}_L : \text{IFin}_L(R*G) \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$ satisfies the following properties:*

- (1) ent_L is upper continuous;
- (2) $\text{ent}_L(R*G \otimes_R K) = L(K)$ for any L -finite left R -module K ;
- (3) $\text{ent}_L(N) = 0$ if and only if $L(N) = 0$, for any $R*G$ -submodule $N \leq R*G \otimes_R K$;
- (4) ent_L is additive.

In particular, ent_L is a length function on $\text{IFin}_L(R*G)$.

Part (1) will be verified in Subsection 6.1, parts (2) and (3) will be proved in Subsection 6.2 and part (4) will be the main result of Subsection 6.3.

We fix all along this section a ring R , an infinite finitely generated amenable group G , a crossed product $R*G$ and a discrete length function $L : R\text{-Mod} \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$ compatible with $R*G$.

6.1 The algebraic entropy is upper continuous

In this subsection we show that the algebraic L -entropy is an upper continuous invariant. We start with the following lemma that deals with the case when M is generated (as $R*G$ -module) by an L -finite R -submodule K , that is, $M = T_G(\lambda, K)$. In such situation one does not need to take a supremum to compute entropy.

Lemma 6.2. *Let M be a left $R*G$ -module such that $M = T_G(\lambda, K)$ for some $K \in \text{Fin}_L(M)$, then*

$$\text{ent}_L(M) = \text{ent}_L(\lambda, K).$$

Proof. Given a finitely generated R -submodule H of M , we can find a finite subset $e \in F \subseteq G$ such that $H \subseteq T_F(\lambda, K)$. This shows that $\text{ent}_L(\lambda, H) \leq \text{ent}_L(\lambda, T_F(\lambda, K))$. Now notice that, using the Følner condition, we obtain

$$\lim_{n \rightarrow \infty} \frac{|F_n F|}{|F_n|} \leq \lim_{n \rightarrow \infty} \frac{|F_n \cup \bigcup_{f \in F} \partial_F(F_n) f|}{|F_n|} \leq 1 + \lim_{n \rightarrow \infty} \sum_{f \in F} \frac{|\partial_F(F_n) f|}{|F_n|} = 1.$$

On the other hand, $|F_n F|/|F_n| \geq 1$ so $\lim_{n \rightarrow \infty} |F_n F|/|F_n| = 1$. We obtain that

$$\text{ent}_L(\lambda, T_F(\lambda, K)) = \lim_{n \rightarrow \infty} \frac{T_{F_n}(\lambda, T_F(\lambda, K))}{|F_n|} = \lim_{n \rightarrow \infty} \frac{T_{F_n F}(\lambda, K)}{|F_n|} \cdot \frac{|F_n|}{|F_n F|} = \text{ent}_L(\lambda, K),$$

where the last equality comes from the fact that $\{F_n F\}_{n \in \mathbb{N}}$ is a Følner sequence (use the fact that, for any $C \subseteq \mathcal{F}(G)$ and $n \in \mathbb{N}$, one has the inclusion $\partial_C(F_n F) \subseteq \partial_{CF^{-1}}(F_n)$ and apply the Følner condition for $\{F_n\}_{n \in \mathbb{N}}$) and the definition of ent_L does not depend on the choice of a particular Følner sequence. Thus, $\text{ent}_L(\lambda, H) \leq \text{ent}_L(\lambda, K)$ for any finitely generated $H \in \mathcal{L}(M)$; one concludes using Proposition 5.6. \square

The upper continuity of ent_L can now be verified easily using the above lemma and Proposition 5.6:

Corollary 6.3. $\text{ent}_L : \text{IFin}_L(R*G) \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$ is an upper continuous invariant.

Proof. The fact that ent_L is an invariant can be derived by the definition and the fact that L is an invariant. Now, let $M \in \text{IFin}_L(R*G)$, then by Proposition 5.6 and Lemmas 6.2 and 4.8 we get

$$\begin{aligned} \text{ent}_L(R*G M) &= \sup\{\text{ent}_L(\lambda, K) : K \text{ finitely generated } R\text{-submodule of } M\} \\ &= \sup\{\text{ent}_L(R*G(T_G(\lambda, K))) : K \text{ finitely generated } R\text{-submodule of } M\} \\ &= \sup\{\text{ent}_L(R*G N) : N \text{ finitely generated } R*G\text{-submodule of } M\}. \end{aligned}$$

\square

6.2 Values on (sub)shifts

This subsection is devoted to compute the values of the algebraic entropy on the $R*G$ -modules of the form $M = R*G \otimes_R K$, for some left R -module K , and their $R*G$ -submodules. Indeed, fix a left R -module K and let $M = R*G \otimes_R K$. Notice that, there is a direct sum decomposition, as a left R -module, ${}_R M \cong \bigoplus_{g \in G} {}_R gK$. We denote the action of G on $\text{Fin}_L(M)$ by

$$\beta : G \rightarrow \text{Aut}(\text{Fin}_L(M)).$$

The choice of the greek letter β to represent this action comes from the *Bernoulli shifts* which are defined in ergodic theory and can be viewed as dual to the actions described here. Notice that the action β has the following properties, where F is a subset of G :

$$T_F(\beta, \underline{g}K) = \bigoplus_{h \in F} h\underline{g}K \quad \text{and} \quad T_G(\beta, \underline{g}K) = M. \quad (6.1)$$

Definition 6.4. A left $R*G$ -module of the form $M = R*G \otimes_R K$ is said to be a Bernoulli shift while any of its $R*G$ -submodules is a subshift.

In the following example we compute the algebraic entropy of Bernoulli shifts.

Example 6.5. In the above notation, suppose $L(K) < \infty$. By Lemma 6.2 and (6.1), we obtain that $\text{ent}_L({}_R M) = \text{ent}_L(\beta, K)$. Furthermore, again by (6.1), $L(T_F(\beta, K))/|F| = L(K)$, for all $F \in \mathcal{F}(G)$. Therefore, $\text{ent}_L(M) = L(K)$.

The computation in the above example shows that the entropy of $M = R*G \otimes_R K$ is 0 if and only if $L(K) = 0$, if and only if $L(M) = 0$. Our next goal is to show that, if ${}_R M$ is a subshift of M , then $\text{ent}_L({}_R M) = 0$ if and only if $L({}_R M) = 0$. This will be proved in Proposition 6.8 but first we need to recall some useful terminology and results from [3].

Definition 6.6. Let E and F be subsets of G . A subset $\mathcal{N} \subseteq G$ is an (E, F) -net if it satisfies the following conditions:

- (1) the subsets $(gE)_{g \in \mathcal{N}}$ are pairwise disjoint, that is, $gE \cap g'E = \emptyset$ for all $g \neq g' \in \mathcal{N}$;
- (2) $G = \bigcup_{g \in \mathcal{N}} gF$.

The following lemma is a variation of [3, Lemmas 2.2 and 4.3].

Lemma 6.7. Let $\{F_n\}_{n \in \mathbb{N}}$ be a Følner sequence, $E \subseteq F \subseteq G$ be finite subsets with $e \in F$ and \mathcal{N} an (E, F) -net. Then,

- (1) there exists an (E, EE^{-1}) -net;
- (2) there exist $\alpha \in (0, 1]$ and $n_0 \in \mathbb{N}$ such that $|F_n \cap \mathcal{N}| \geq \alpha \cdot |F_n|$, for all $n > n_0$.

Proof. Part (1) is proved in [3, Lemma 2.2], so let us concentrate on part (2). For each $n \in \mathbb{N}$, let $F_n^{+F} = \text{Out}_F(F_n) \cap \mathcal{N}$ and notice that $F_n^{+F} \setminus (F_n \cap \mathcal{N}) \subseteq \partial_F(F_n)$. Furthermore, since F_n is covered by the sets gF , $g \in F_n^{+F}$, we have $|F_n| \leq |F| |F_n^{+F}|$. Let now $\alpha_1 = 1/|F|$, thus

$$\alpha_1 |F_n| - |F_n \cap \mathcal{N}| \leq |F_n^{+F}| - |F_n \cap \mathcal{N}| \leq |F_n^{+F} \setminus (F_n \cap \mathcal{N})| \leq \partial_F(F_n).$$

Let $\alpha_2 \in (0, \alpha_1)$. By the Følner condition, there exists $n_0 \in \mathbb{N}$ such that $|\partial_F(F_n)|/|F_n| \leq \alpha_2$ for all $n > n_0$. Thus, letting $\alpha = \alpha_1 - \alpha_2 \in (0, 1]$, we get $|F_n \cap \mathcal{N}| \geq \alpha_1 |F_n| - \partial_F(F_n) \geq \alpha |F_n|$, for all $n > n_0$. \square

Proposition 6.8. Let K be an L -finite left R -module and ${}_R M$ be a subshift of $M = R*G \otimes_R K$. Then,

$$\text{ent}_L({}_R M) = 0 \quad \text{if and only if} \quad L({}_R M) = 0.$$

Proof. Suppose $L({}_R M) \neq 0$, then there exists $x \in M$ such that $L(Rx) \neq 0$. Let E be the set of all elements $h \in G$ such that, writing $x = \sum_{g \in G} g x_g$, the component x_h is not 0. We fix an (E, EE^{-1}) -net \mathcal{N} . Notice that, given $f_1 \neq f_2 \in \mathcal{N}$, then $\beta_{f_1}(Rx) \cap \beta_{f_2}(Rx) = \emptyset$. Thus, by Lemma 6.7 we can find $n_0 \in \mathbb{N}$ and $\alpha \in (0, 1)$ such that

$$L(T_{F_n}(\beta, Rx)) \geq L(T_{F_n \cap \mathcal{N}}(\beta, Rx)) = |F_n \cap \mathcal{N}| L(Rx) \geq \alpha |F_n| L(Rx)$$

for all $n > n_0$. In particular, $\text{ent}_L({}_R M) \geq \text{ent}_L(\beta, Rx) \geq \alpha L(Rx) \neq 0$. \square

6.3 The Addition Theorem

In this subsection we complete the proof of Theorem 6.1 verifying a very strong property of the algebraic entropy, that is, its additivity. In particular, we have to verify that, given a locally L -finite left $R*G$ -module M , and an $R*G$ -submodule $N \leq M$,

$$\text{ent}_L(R*GM) = \text{ent}_L(R*GN) + \text{ent}_L(R*G(M/N)). \quad (6.2)$$

We fix all along this subsection the following notations for the actions induced by the $R*G$ -module structures:

$$\begin{aligned} \lambda : G &\rightarrow \text{Aut}(\text{Fin}_L(M)) & \lambda' : G &\rightarrow \text{Aut}(\text{Fin}_L(N)) & \bar{\lambda} : G &\rightarrow \text{Aut}(\text{Fin}_L(M/N)) \\ g &\mapsto \lambda_g & g &\mapsto \lambda'_g = \lambda_g \upharpoonright_{\text{Fin}_L(N)} & g &\mapsto \bar{\lambda}_g \end{aligned}$$

We start proving the inequality “ \geq ” of (6.2).

Lemma 6.9. $\text{ent}_L(R*GM) \geq \text{ent}_L(R*GN) + \text{ent}_L(R*G(M/N)).$

Proof. Let $K_1 \leq N$ and $K_2 \leq M/N$ be finitely generated R -submodules. Fix a finitely generated submodule $K \leq M$ such that $(K + N)/N = K_2$ and $K \cap N \supseteq K_1$. Given a finite subset $F \subseteq G$, by Lemma 4.5 there is a short exact sequence

$$0 \rightarrow T_F(\lambda, K) \cap N \rightarrow T_F(\lambda, K) \rightarrow T_F(\bar{\lambda}, K_2) \rightarrow 0.$$

Noticing that $T_F(\lambda', K_1) \subseteq T_F(\lambda, K) \cap N$, we get $L(T_F(\lambda, K)) \geq L(T_F(\lambda', K_1)) + L(T_F(\bar{\lambda}, K_2))$. Applying this inequality to the sets belonging to a Følner sequence $\{F_n\}_{n \in \mathbb{N}}$, yields

$$\text{ent}_L(R*GM) \geq \text{ent}_L(\lambda, K) \geq \text{ent}_L(\lambda', K_1) + \text{ent}_L(\bar{\lambda}, K_2).$$

The result follows by the arbitrariness of the choice of K_1 and K_2 . \square

The first step in proving the converse inequality is to show that we can reduce the problem to the case when both M and N are finitely generated $R*G$ -modules. This goal is obtained in the following corollary (which is just a reformulation of Corollary 4.5), and the subsequent two lemmas.

Corollary 6.10. *Let $\{F_n\}_{n \in \mathbb{N}}$ be a Følner exhaustion of G . Then, for any $\varepsilon \in (0, 1/4)$ and $n \in \mathbb{N}$ there exist $n_1, \dots, n_k \in \mathbb{N}$ with $n \leq n_1 \leq \dots \leq n_k$ such that, given an L -finite submodule $K \leq M$,*

$$\text{ent}_L(\lambda, K) \leq \varepsilon \cdot L(K) + \frac{1}{1 - \varepsilon} \cdot \max_{1 \leq i \leq k} \frac{L(T_{F_{n_i}}(\lambda, K))}{|F_{n_i}|}.$$

Proof. This is a straightforward application of Corollary 4.5. In fact, the function $f_K : \mathcal{F}(G) \rightarrow \mathbb{R}_{\geq 0}$ such that $f_K(F) = L(T_F(\lambda, K))$ satisfies the hypotheses of such corollary for any L -finite R -submodule K of M , by Lemma 5.2. Furthermore, $\lim_{n \rightarrow \infty} f_K(F_n)/|F_n| = \text{ent}_L(\lambda, K)$ by the definition of entropy. \square

Lemma 6.11. *Consider a sequence $N_0 \leq N_1 \leq \dots \leq N_t \leq \dots \leq M$ of $R*G$ -submodules of M such that $N = \bigcup_{t \in \mathbb{N}} N_t$ and let $\bar{\lambda}_t : G \rightarrow \text{Aut}(\text{Fin}_L(M/N_t))$ be the actions induced on the quotients. Then, given an L -finite submodule $K \leq M$ and letting $\bar{K} = (K + N)/N$ and $\bar{K}_t = (K + N_t)/N_t$ for all $t \in \mathbb{N}$,*

$$\text{ent}_L(\bar{\lambda}, \bar{K}) = \inf_{t \in \mathbb{N}} \text{ent}_L(\bar{\lambda}_t, \bar{K}_t).$$

Proof. The inequality “ \leq ” follows by Lemma 5.7. On the other hand, for all $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that

$$\frac{L(T_{F_n}(\bar{\lambda}, \bar{K}))}{|F_n|} \leq \text{ent}_L(\bar{\lambda}, \bar{K}) + \varepsilon, \quad \text{for all } n \geq n_\varepsilon. \quad (6.3)$$

By Corollary 6.10, for any given $\varepsilon' \in (0, 1/4)$, there exist $n_1, \dots, n_k \in \mathbb{N}$ such that $n_\varepsilon \leq n_1 \leq \dots \leq n_k$ and

$$\text{ent}_L(\bar{\lambda}_t, \bar{K}_t) \leq \varepsilon' \cdot L(\bar{K}_t) + \frac{1}{1 - \varepsilon'} \cdot \max_{1 \leq i \leq k} \frac{L(T_{F_{n_i}}(\bar{\lambda}_t, \bar{K}_t))}{|F_{n_i}|} \quad (6.4)$$

holds for all $t \in \mathbb{N}$. Now, for all $i \in \{1, \dots, k\}$, the set $\{L(T_{F_{n_i}}(\bar{\lambda}_t, \bar{K}_t)) : t \in \mathbb{N}\}$ is a set of values of L , all smaller than or equal to the finite value $L(T_{F_i}(\lambda, K))$. Since we supposed L to be discrete, this set has a minimum, say $L(T_{F_{n_i}}(\bar{\lambda}_{t_i}, \bar{K}_{t_i}))$. Let $s = \max_{1 \leq i \leq k} t_i$ and notice that

$$L(T_{F_{n_i}}(\bar{\lambda}_s, \bar{K}_s)) = L(T_{F_{n_i}}(\bar{\lambda}, \bar{K})), \quad \text{for all } i = 1, \dots, k.$$

(This follows by the additivity of L and the fact that $L((N \cap (T_{F_{n_i}}(\lambda, K) + N_s))/N_s) = 0$, as this module is the union of modules of the form $(N_t \cap (T_{F_{n_i}}(\lambda, K) + N_s))/N_s$ with $t \geq s$, which are L -singular by the choice of s).

Using the above computations, we get

$$\begin{aligned} \inf_{t \in \mathbb{N}} \text{ent}_L(\bar{\lambda}_t, \bar{K}_t) &\leq \text{ent}_L(\bar{\lambda}_s, \bar{K}_s) \leq \varepsilon' \cdot L(\bar{K}_s) + \frac{1}{1 - \varepsilon'} \cdot \max_{1 \leq i \leq k} \frac{L(T_{F_{n_i}}(\bar{\lambda}_s, \bar{K}_s))}{|F_{n_i}|} \\ &= \varepsilon' \cdot L(\bar{K}_s) + \frac{1}{1 - \varepsilon'} \cdot \max_{1 \leq i \leq k} \frac{L(T_{F_{n_i}}(\bar{\lambda}, \bar{K}))}{|F_{n_i}|} \\ &\leq \varepsilon' \cdot L(K) + \frac{1}{1 - \varepsilon'} \cdot (\text{ent}_L(\bar{\lambda}, \bar{K}) + \varepsilon). \end{aligned}$$

Letting ε' tend to 0 we obtain that $\inf_{t \in \mathbb{N}} \text{ent}_L(\bar{\lambda}_t, K) \leq \text{ent}_L(\bar{\lambda}, \bar{K}) + \varepsilon$. As this holds for all $\varepsilon > 0$, the conclusion follows. \square

As we announced, we can now prove the following reduction to the case when $R_*G N$ and $R_*G M$ are finitely generated.

Lemma 6.12. *If (6.2) holds whenever $R_*G N$ and $R_*G M$ are finitely generated, then it holds in general.*

Proof. We already proved that the inequality “ \geq ” in (6.2) always holds, thus we concentrate just on the converse inequality. Indeed, given a finitely generated submodule $R_*G K$ of $R_*G M$ we claim that

$$\text{ent}_L(R_*G K) = \text{ent}_L(R_*G(K \cap N)) + \text{ent}_L(R_*G((K + N)/N)). \quad (6.5)$$

Notice that, if we prove the above claim, we can easily conclude using upper continuity as follows:

$$\begin{aligned} \text{ent}_L(R_*G M) &= \sup\{\text{ent}_L(R_*G K) : K \leq M \text{ f.g.}\} \\ &= \sup\{\text{ent}_L(R_*G(K \cap N)) + \text{ent}_L(R_*G(K + N)/N) : K \leq M \text{ f.g.}\} \\ &\leq \sup\{\text{ent}_L(R_*G(K \cap N)) : K \leq M \text{ f.g.}\} + \sup\{\text{ent}_L(R_*G(K + N)/N) : K \leq M \text{ f.g.}\} \\ &= \text{ent}_L(R_*G N) + \text{ent}_L(R_*G(M/N)). \end{aligned}$$

It remains to verify claim (6.5). Indeed, choose a finitely generated R -submodule H of K , such that $K = T_G(\lambda, H)$. Notice also that $K \cap N = \bigcup_{n \in \mathbb{N}} T_G(\lambda, T_{N_n(B)}(\lambda, H) \cap N)$. For all $n \in \mathbb{N}$, we let H_n be an L -essential, finitely generated R -submodule of $T_{N_n(B)}(\lambda, H) \cap N$. By Proposition 5.6, we obtain that

$$\text{ent}_L(R_*G(N \cap K)) = \sup_{n \in \mathbb{N}} \text{ent}_L(\lambda, T_{N_n(B)}(\lambda, H) \cap N) = \sup_{n \in \mathbb{N}} \text{ent}_L(\lambda, H_n).$$

We let $K' = \bigcup_{n \in \mathbb{N}} T_G(\lambda, H_n)$. Notice that K' is L -essential in $K \cap N$ (in fact, $(K \cap N)/K'$ is the union of modules of the form $((T_{N_n(B)}(\lambda, H) \cap N) + K')/K'$ and each of these modules is a quotient of an L -singular module of the form $(T_{N_n(B)}(\lambda, H) \cap N)/T_G(\lambda, H_n)$). By Lemma 5.8 and Proposition 5.6 we obtain that

$$\text{ent}_L(R_*G(K \cap N)) = \text{ent}_L(R_*G K') = \lim_{n \rightarrow \infty} \text{ent}_L(R_*G T_G(\lambda, H_n)) = \sup_{n \in \mathbb{N}} \text{ent}_L(R_*G T_G(\lambda, H_n)).$$

Similarly, one derives by Lemma 6.11 that

$$\text{ent}_L(R_*G(K/(K \cap N))) = \text{ent}_L(R_*G(K/K')) = \lim_{n \rightarrow \infty} \text{ent}_L(R_*G(K/T_G(\lambda, H_n))) = \inf_{n \in \mathbb{N}} \text{ent}_L(R_*G(K/T_G(\lambda, H_n))).$$

By hypothesis, $\text{ent}_L(R_*G T_G(\lambda, H_n)) + \text{ent}_L(R_*G(K/T_G(\lambda, H_n))) = \text{ent}_L(R_*G K)$ for all $n \in \mathbb{N}$. Putting together all these computations we obtain:

$$\begin{aligned} \text{ent}_L(R_*G K) &= \lim_{n \rightarrow \infty} (\text{ent}_L(R_*G T_G(\lambda, H_n)) + \text{ent}_L(R_*G(K/T_G(\lambda, H_n)))) \\ &= \lim_{n \rightarrow \infty} \text{ent}_L(R_*G T_G(\lambda, H_n)) + \lim_{n \rightarrow \infty} \text{ent}_L(R_*G(K/T_G(\lambda, H_n))) \\ &= \text{ent}_L(R_*G(K \cap N)) + \text{ent}_L(R_*G(K/(K \cap N))), \end{aligned}$$

which verifies (6.5), concluding the proof. \square

Finally, we have all the instruments to conclude the proof of the additivity of ent_L . The computations in the proof of the following lemma are freely inspired to the proof of the Abramov-Rokhlin Formula given in [37]. The context (and even the statements) in that paper is quite different but the ideas contained there can be perfectly adapted to our needs.

Lemma 6.13. $\text{ent}_L(R*G M) \leq \text{ent}_L(R*G N) + \text{ent}_L(R*G(M/N))$.

Proof. By Lemma 6.12, we can suppose that both M and N are finitely generated $R*G$ -modules. In particular, there exists a finitely generated R -submodule $K' \leq N$ and a finitely generated R -submodule $\bar{K}_2 \leq M/N$ such that $N = T_G(\lambda', K')$ and $M/N = T_G(\bar{\lambda}, \bar{K}_2)$. Since \bar{K}_2 is finitely generated, there exists a finitely generated R -submodule K_2 of M , such that $(K_2 + N)/N = \bar{K}_2$. We let $K = K' + K_2$ and we notice that $M = T_G(\lambda, K)$. Finally, we let K_1 be an L -essential finitely generated submodule of $K \cap N$ containing K' . Notice that, by Lemma 6.2 we obtain that $\text{ent}_L(R*G M) = \text{ent}_L(\lambda, K)$, $\text{ent}_L(R*G N) = \text{ent}_L(\lambda', K \cap N) = \text{ent}_L(\lambda', K_1)$ and $\text{ent}_L(R*G(M/N)) = \text{ent}_L(\bar{\lambda}, \bar{K}_2)$.

Let $\varepsilon \in (0, 1/4)$ and fix a Følner exhaustion $\{F_n\}_{n \in \mathbb{N}}$. By the existence of the limits defining the algebraic L -entropies, we can find $\bar{n} \in \mathbb{N}$ such that, for all $n > \bar{n}$

$$\left| \frac{L(T_{F_n}(\lambda, K))}{|F_n|} - \text{ent}_L(M) \right| < \varepsilon, \quad \left| \frac{L(T_{F_n}(\lambda', K_1))}{|F_n|} - \text{ent}_L(N) \right| < \varepsilon, \quad \left| \frac{L(T_{F_n}(\bar{\lambda}, \bar{K}_2))}{|F_n|} - \text{ent}_L(M/N) \right| < \varepsilon. \quad (6.6)$$

For all $m \in \mathbb{N}$,

$$\frac{L(T_{F_m}(\lambda, K))}{|F_m|} = \frac{L(T_{F_m}(\lambda', K_1))}{|F_m|} + \frac{L(T_{F_m}(\lambda, K)/T_{F_m}(\lambda', K_1))}{|F_m|}$$

and so, for all $m \geq \bar{n}$,

$$\begin{aligned} \text{ent}_L(M) &< \frac{L(T_{F_m}(\lambda, K))}{|F_m|} + \varepsilon = \frac{L(T_{F_m}(\lambda', K_1))}{|F_m|} + \frac{L(T_{F_m}(\lambda, K)/T_{F_m}(\lambda', K_1))}{|F_m|} + \varepsilon \\ &< \text{ent}_L(N) + \frac{L(T_{F_m}(\lambda, K)/T_{F_m}(\lambda', K_1))}{|F_m|} + 2\varepsilon. \end{aligned} \quad (6.7)$$

In the remaining part of the proof we are going to show that there exists a positive integer k such that

$$\frac{L(T_{F_m}(\lambda, K)/T_{F_m}(\lambda', K_1))}{|F_m|} \leq \frac{1}{1-\varepsilon} \text{ent}_L(R*G(M/N)) + \varepsilon \left(L(K)(k+1) + \frac{1}{1-\varepsilon} \right), \quad (6.8)$$

for all big enough $m \in \mathbb{N}$. Applying (6.8) to (6.7), one gets

$$\text{ent}_L(M) < \text{ent}_L(N) + \frac{1}{1-\varepsilon} \text{ent}_L(M/N) + \varepsilon \left(L(K)(k+1) + \frac{1}{1-\varepsilon} \right) + 2\varepsilon$$

which, as it holds for all $\varepsilon \in (0, 1/4)$, gives the desired inequality. Thus, to conclude we have to verify (6.8).

Since $\{F_n\}$ is a Følner exhaustion, we have that $N = \bigcup_{n \in \mathbb{N}} T_{F_n}(\lambda', K_1)$ and so, for any L -finite submodule $H \leq M$, we can use the upper continuity of L to obtain that $L(H \cap N) = \lim_{n \rightarrow \infty} L(H \cap T_{F_n}(\lambda', K_1))$. Now, by additivity, this implies that $L((H+N)/N) = \lim_{n \rightarrow \infty} L((H+T_{F_n}(\lambda', K_1))/T_{F_n}(\lambda', K_1))$ and, by the discreteness of L , this limit is the minimum of the values. By Theorem 4.3, there exist $\bar{n} < n_1 < \dots < n_k \in \mathbb{N}$ such that $\{F_{n_1}, \dots, F_{n_k}\}$ ε -quasi-tiles F_m for all $m \geq \bar{n}$. Applying the above argument with $H = T_{F_{n_i}}(\lambda, K)$ (for all $i = 1, \dots, k$), we can find $\bar{\bar{n}} \in \mathbb{N}$ such that

$$L \left(\frac{T_{F_{n_i}}(\lambda, K) + T_{F_{n_i}}(\lambda', K_1)}{T_{F_{n_i}}(\lambda', K_1)} \right) = L \left(\frac{T_{F_{n_i}}(\lambda, K) + N}{N} \right) \quad (6.9)$$

for all $n \geq \bar{\bar{n}}$ and all $i = 1, \dots, k$. From now on we suppose that m is a positive integer such that

$$m \geq \max\{\bar{n}, \bar{\bar{n}}\} \quad \text{and} \quad |\partial_{F_m}(F_m)|/|F_m| \leq \varepsilon. \quad (6.10)$$

(Where the second condition can be assumed since $\{F_n\}_{n \in \mathbb{N}}$ is Følner.) As $\{F_{n_1}, \dots, F_{n_k}\}$ ε -quasi-tiles F_m , we can choose tiling centers C_1, \dots, C_k . We obtain the following inequalities

$$|F_m| \geq \left| \bigcup_{i=1}^k C_i F_{n_i} \right| \geq \max \left\{ (1-\varepsilon)|F_m|, (1-\varepsilon) \sum_{i=1}^k |C_i| |F_{n_i}| \right\}, \quad (6.11)$$

which imply that

$$\frac{L(T_{F_m \setminus \bigcup_{i=1}^k C_i F_{n_i}}(\lambda, K))}{|F_m|} \leq \frac{|F_m \setminus \bigcup_{i=1}^k C_i F_{n_i}|}{|F_m|} L(K) = \left(1 - \frac{|\bigcup_{i=1}^k C_i F_{n_i}|}{|F_m|} \right) L(K) \leq \varepsilon L(K).$$

Applying this computation and using again (6.11), one gets:

$$\begin{aligned} \frac{L(T_{F_m}(\lambda, K)/T_{F_m}(\lambda', K_1))}{|F_m|} &\leq \frac{1}{|\bigcup_{i=1}^k C_i F_{n_i}|} L\left(\frac{T_{\bigcup_{i=1}^k C_i F_{n_i}}(\lambda, K) + T_{F_m}(\lambda', K_1)}{T_{F_m}(\lambda', K_1)}\right) + \varepsilon L(K) \\ &\leq \frac{(1-\varepsilon)^{-1}}{\sum_{i=1}^k |C_i||F_{n_i}|} \sum_{i=1}^k L\left(\frac{T_{C_i F_{n_i}}(\lambda, K) + T_{F_m}(\lambda', K_1)}{T_{F_m}(\lambda', K_1)}\right) + \varepsilon L(K). \end{aligned} \quad (6.12)$$

Now, let $t_i = (|C_i||F_{n_i}|)/\sum_{j=1}^k |C_j||F_{n_j}|$ and notice that $t_i \in (0, 1)$ and $\sum_{i=1}^k t_i = 1$. Then

$$\frac{1}{\sum_{j=1}^k |C_j||F_{n_j}|} \sum_{i=1}^k L\left(\frac{T_{C_i F_{n_i}}(\lambda, K) + T_{F_m}(\lambda', K_1)}{T_{F_m}(\lambda', K_1)}\right) = \sum_{i=1}^k \frac{t_i}{|C_i||F_{n_i}|} L\left(\frac{T_{C_i F_{n_i}}(\lambda, K) + T_{F_m}(\lambda', K_1)}{T_{F_m}(\lambda', K_1)}\right). \quad (6.13)$$

Notice that, since $\{F_n\}_{n \in \mathbb{N}}$ is a Følner exhaustion, then $1 \in F_{n_i}$ and so $C_i \subseteq F_m \subseteq \text{In}_{F_{\overline{n}}}(F_m) \cup \partial_{F_{\overline{n}}}(F_m)$ for all $i = 1, \dots, k$. Thus,

$$\begin{aligned} \frac{1}{|C_i||F_{n_i}|} L\left(\frac{T_{C_i F_{n_i}}(\lambda, K) + T_{F_m}(\lambda', K_1)}{T_{F_m}(\lambda', K_1)}\right) &\leq \frac{1}{|C_i||F_{n_i}|} \sum_{c \in C_i} L\left(\frac{T_{c F_{n_i}}(\lambda, K) + T_{F_m}(\lambda', K_1)}{T_{F_m}(\lambda', K_1)}\right) \\ &\leq \frac{1}{|C_i||F_{n_i}|} \left(\sum_{c \in C_i \cap \text{In}_{F_{\overline{n}}}(F_m)} L\left(\frac{T_{F_{n_i}}(\lambda, K) + T_{c^{-1}F_m}(\lambda', K_1)}{T_{c^{-1}F_m}(\lambda', K_1)}\right) + |\partial_{F_{\overline{n}}}(F_m)|L(K) \right) \\ &\leq \frac{1}{|F_{n_i}|} L\left(\frac{T_{F_{n_i}}(\lambda, K) + T_{F_{\overline{n}}}(\lambda', K_1)}{T_{F_{\overline{n}}}(\lambda', K_1)}\right) + \frac{|\partial_{F_{\overline{n}}}(F_m)|L(K)}{|C_i||F_{n_i}|} = \frac{1}{|F_{n_i}|} L\left(\frac{T_{F_{n_i}}(\lambda, K) + N}{N}\right) + \frac{|\partial_{F_{\overline{n}}}(F_m)|L(K)}{|C_i||F_{n_i}|} \end{aligned} \quad (6.14)$$

where the first inequality in the last line comes from the fact that (by definition of $\text{In}_{F_{\overline{n}}}(F_m)$), $F_{\overline{n}} \subseteq c^{-1}F_m$ for all $c \in C_i \cap \text{In}_{F_{\overline{n}}}(F_m)$; the last equality is an application of (6.9)). Let us assemble together the above computations:

$$\begin{aligned} \frac{L(T_{F_m}(\lambda, K)/T_{F_m}(\lambda', K_1))}{|F_m|} &\stackrel{(6.12)}{\leq} \frac{(1-\varepsilon)^{-1}}{\sum_{i=1}^k |C_i||F_{n_i}|} \sum_{i=1}^k L\left(\frac{T_{C_i F_{n_i}}(\lambda, K) + T_{F_m}(\lambda', K_1)}{T_{F_m}(\lambda', K_1)}\right) + \varepsilon L(K) \\ &\stackrel{(6.13)}{\leq} \frac{1}{1-\varepsilon} \sum_{i=1}^k \frac{t_i}{|C_i||F_{n_i}|} L\left(\frac{T_{C_i F_{n_i}}(\lambda, K) + T_{F_m}(\lambda', K_1)}{T_{F_m}(\lambda', K_1)}\right) + \varepsilon L(K) \\ &\stackrel{(6.14)}{\leq} \frac{1}{1-\varepsilon} \sum_{i=1}^k \frac{t_i}{|F_{n_i}|} L\left(\frac{T_{F_{n_i}}(\lambda, K) + N}{N}\right) + \sum_{i=1}^k \frac{t_i}{1-\varepsilon} \frac{|\partial_{F_{\overline{n}}}(F_m)|L(K)}{|C_i||F_{n_i}|} + \varepsilon L(K) \\ &\leq \sum_{i=1}^k \frac{t_i}{1-\varepsilon} \frac{T_{F_{n_i}}(\lambda, K)}{|F_{n_i}|} + \frac{1}{1-\varepsilon} \sum_{i=1}^k \frac{|\partial_{F_{\overline{n}}}(F_m)|L(K)}{\sum_{i=1}^k |C_i||F_{n_i}|} + \varepsilon L(K) \\ &\stackrel{(6.11), (6.10), (6.6)}{\leq} \frac{1}{1-\varepsilon} (\text{ent}_L(R * G(M/N)) + \varepsilon) + \varepsilon L(K)(k+1) \\ &= \frac{1}{1-\varepsilon} \text{ent}_L(R * G(M/N)) + \varepsilon \left(L(K)(k+1) + \frac{1}{1-\varepsilon} \right). \end{aligned}$$

□

7 Applications

7.1 Stable finiteness

Recall that a *concrete category* is a pair $(\mathfrak{C}, F : \mathfrak{C} \rightarrow \text{Set})$, where \mathfrak{C} is a category and F is a faithful functor to the category of sets. In a concrete category one says that a morphism ϕ is injective (resp., surjective, bijective) if so is the map $F(\phi)$. An object X of \mathfrak{C} is said to be *hopfian* if any surjective endomorphism of X is bijective. Such notion is usually introduced in categories of (Abelian) groups, rings, modules, or topological spaces.

Our interest in hopficity, comes from the following observation. Recall that a ring R is *directly finite* if $xy = 1$ implies $yx = 1$ for all $x, y \in R$. Furthermore, R is *stably finite* if the ring $\text{Mat}_k(R)$ of $k \times k$ square matrices with coefficients in R , is directly finite for all $k \in \mathbb{N}_+$.

Lemma 7.1. *Given a ring R and a positive integer k , the following are equivalent:*

- (1) R_R^k is hopfian (as a right R -module);
- (2) ${}_R R^k$ is hopfian (as a left R -module);
- (3) $\text{Mat}_k(R)$ is directly finite.

A long-standing open question about directly finite rings is the following conjecture due to Kaplansky

Conjecture 7.2 (Kaplansky). *For any field \mathbb{K} and any group G , the group ring $\mathbb{K}[G]$ is stably finite.*

The above conjecture was verified by Kaplansky [20] in case $\mathbb{K} = \mathbb{C}$. Furthermore, it was proved by Ara, O'Meara and Perera [2] that $\mathbb{K}[G]$ is stably finite whenever G is residually amenable and \mathbb{K} is an arbitrary division ring. This last result was generalized by Elek and Szabó [11], who proved the following

Theorem 7.3 (Elek and Szabó). *For any division ring \mathbb{K} and any sofic group G , the group ring $\mathbb{K}[G]$ is stably finite.*

Notice also that Ara et al. proved that any crossed product $\mathbb{K} * G$ is stably finite, provided \mathbb{K} is a division ring and G is an amenable group.

A straightforward consequence of the above theorem is that $\text{Mat}_n(\mathbb{K}[G])$ is stably finite for any division ring \mathbb{K} and any sofic group G . Now, by the Artin-Wedderburn Theorem, given a semisimple Artinian ring R , there exist positive integers $k, n_1, \dots, n_k \in \mathbb{N}_+$ and division rings $\mathbb{K}_1, \dots, \mathbb{K}_k$ such that $R \cong \text{Mat}_{n_1}(\mathbb{K}_1) \times \dots \times \text{Mat}_{n_k}(\mathbb{K}_k)$. This implies that, $R[G] \cong \text{Mat}_{n_1}(\mathbb{K}_1[G]) \times \dots \times \text{Mat}_{n_k}(\mathbb{K}_k[G])$, thus a consequence of the above theorem is that $R[G]$ is stably finite whenever R is semisimple Artinian and G sofic. This result can be further generalized as follows:

Remark 7.4. [Ferran Cedó, private communication (2012)] If R is a ring with left Krull dimension (for example see [24]), then $R[G]$ is stably finite. First of all, notice that, if I is a nilpotent ideal of R , then $I[G] = R[G]I$ is a nilpotent ideal of $R[G]$ and so one can reduce the problem modulo nilpotent ideals. Now, by [24, Corollary 6.3.8], the prime radical N of R is nilpotent and $N = P_1 \cap \dots \cap P_m$, where P_1, \dots, P_m are minimal prime ideals. Thus, by [24, Proposition 6.3.5], R/N is a semiprime Goldie ring and so, by [24, Theorem 2.3.6] R/N has a classical semisimple Artinian ring of quotients S . In particular, $(R/N)[G]$ embeds in $S[G]$ and it is therefore stably finite.

Both the proof of the residually amenable case due to Ara, O'Meara and Perera, and the proof of the sofic case due to Elek and Szabó, consist in finding a suitable embedding of $\mathbb{K}[G]$ in a ring which is known to be stably finite. Such methods are really effective but, as far as we know, cannot be used to obtain information on the modules over $\mathbb{K}[G]$. It seems natural to ask the following

Question 7.5. *Let G be a group, R a ring and let $R * G$ be a fixed crossed product. Given a finitely generated left $R * G$ -module ${}_R * G M$, when is ${}_R * G M$ an hopfian module?*

By Lemma 7.1, we know that Theorem 7.3 means exactly that, when R is a division ring and G is sofic, then any free left $R[G]$ -module of finite rank is hopfian. This is true more generally for finitely generated projective modules:

Lemma 7.6. *Let R be a ring, let ${}_R M$ and ${}_R N$ be left R -modules; suppose that ${}_R M$ is hopfian and ${}_R N$ is a direct summand of ${}_R M$. Then ${}_R N$ is hopfian as well.*

In particular, R is stably finite (if and) only if any finitely generated projective left R -module is hopfian.

Proof. Let ${}_R N' \leq M$ be a complement for N , that is $M \cong N \oplus N'$ and let $\phi : N \rightarrow N$ be a surjective endomorphism. Let $\Phi : M \rightarrow M$ be such that $\Phi(n, n') = (\phi(n), n')$, for all $n \in N$ and $n' \in N'$. Clearly Φ is surjective and $\text{Ker}(\Phi) = \text{Ker}(\phi) \oplus \{0\}$. Now, $\text{Ker}(\Phi)$ is trivial by the hopficity of M and so also $\text{Ker}(\phi) = 0$ concluding the proof. \square

Notice that in general the class of hopfian modules is not closed under taking finite direct sums, for a classical (counter)example see [7, Example 3].

In [16], Anna Giordano Bruno and the author studied the concept of hereditarily hopfian Abelian group. Generalizing that situation, we say that a left R -module M over a ring R is *hereditarily hopfian* if and only if all its submodules are hopfian.

Example 7.7. The Abelian group of p -adic integers \mathbb{J}_p is hopfian but not hereditarily hopfian.

In fact, any \mathbb{Z} -linear endomorphism of \mathbb{J}_p is also \mathbb{J}_p -linear, so $\text{End}_{\mathbb{Z}}(\mathbb{J}_p)$ is canonically isomorphic, as a ring, to the commutative ring $\text{End}_{\mathbb{J}_p}(\mathbb{J}_p) \cong \mathbb{J}_p$. Now, if $\phi : \mathbb{J}_p \rightarrow \mathbb{J}_p$ is a surjective endomorphism, as \mathbb{J}_p is projective as \mathbb{J}_p -module, there is $\psi : \mathbb{J}_p \rightarrow \mathbb{J}_p$ such that $\phi\psi = 1$ and so $1 = \phi\psi = \psi\phi$, proving that ϕ is an isomorphism and so \mathbb{J}_p is hopfian.

On the other hand, ${}_Z \mathbb{J}_p$ has infinite torsion-free rank, that is, there is a subgroup $G \leq \mathbb{J}_p$ of the form $G \cong \mathbb{Z}^{(\mathbb{N})}$ which is clearly not hopfian. Hence, ${}_Z \mathbb{J}_p$ is not hereditarily hopfian.

Using the theory of algebraic entropy we can now prove that a large class of left $R*G$ -modules is hereditarily hopfian, in case R is left Noetherian and G amenable. We remark that this is a very strong version of Kaplasky's Stable Finiteness Conjecture in the amenable case which can be re-obtained as a corollary.

Theorem 7.8. *Let R be a left Noetherian ring, G a finitely generated amenable group and let $R*G$ be a fixed crossed product. Then, for any finitely generated left R -module ${}_R K \in R\text{-Mod}$, the left $R*G$ -module $R*G \otimes_R K$ is hereditarily hopfian.*

*In particular, $\text{End}_{R*G}(M)$ is stably finite for any submodule ${}_R M \leq R*G \otimes_R K$.*

The proof of the above theorem makes use of the full force of the localization techniques introduced in Section 2. Such heavy machinery hides in some sense the idea behind the proof; this is the reason for which we prefer to give first the proof of the following more elementary statement, whose proof is far more transparent.

Lemma 7.9. *Let \mathbb{K} be a division ring, let G be a finitely generated amenable group and fix a crossed product $\mathbb{K}*G$. For all $n \in \mathbb{N}_+$, $(\mathbb{K}*G)^n = \mathbb{K}*G \otimes \mathbb{K}^n$ is a hereditarily hopfian left $\mathbb{K}*G$ -module.*

Proof. Let $n \in \mathbb{N}_+$ and choose $\mathbb{K}*G$ -submodules $N \leq M \leq (\mathbb{K}*G)^n$ such that there exists a short exact sequence

$$0 \rightarrow N \rightarrow M \rightarrow M \rightarrow 0,$$

we have to show that $N = 0$. The length function $\dim : \mathbb{K}\text{-Mod} \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$ is compatible with any crossed product, so we can consider the dim-entropy of left $\mathbb{K}*G$ -modules. In particular, we have that

$$\text{ent}_{\dim}(M) = \text{ent}_{\dim}(M) + \text{ent}_{\dim}(N) \quad \text{and} \quad 0 \leq \text{ent}_{\dim}(N) \leq \text{ent}_{\dim}(M) \leq \text{ent}_{\dim}(\mathbb{K}*G \otimes \mathbb{K}^n) = n.$$

Thus, $\text{ent}_{\dim}(N) = 0$. By Proposition 6.8, this implies that $\dim(N) = 0$, that is, $N = 0$. \square

The same argument of the above proof can be used to prove Theorem 7.8, modulo the fundamental tool of Gabriel dimension:

Proof of Theorem 7.8. Consider a left $R*G$ -submodule $M \leq R*G \otimes_R K$ and a short exact sequence of left $R*G$ -modules

$$0 \rightarrow \text{Ker}(\phi) \rightarrow M \xrightarrow{\phi} M \rightarrow 0.$$

In order to go further with the proof we need to show that, as a left R -module, the Gabriel dimension of $\text{Ker}(\phi)$ is a successor ordinal whenever it is not -1 (i.e., whenever $\text{Ker}(\phi) \neq 0$). This follows by the following

Lemma 7.10. *In the hypotheses of Theorem 7.8, $\text{G.dim}({}_R N)$ is a successor ordinal for any non-trivial R -submodule $N \leq R*G \otimes_R K$.*

Proof. A consequence of Corollary 2.16 is that $\mathbf{T}_{\alpha+1}(K)/\mathbf{T}_{\alpha}(K) \neq 0$ for just finitely many ordinals α . Notice that $\mathbf{T}_{\alpha}(R*G \otimes_R K) \cong R*G \otimes_R \mathbf{T}_{\alpha}(K)$, as left R -modules, for any ordinal α . Thus, $\mathbf{T}_{\alpha+1}(R*G \otimes_R K)/\mathbf{T}_{\alpha}(R*G \otimes_R K) \neq 0$ for finitely many ordinals. Notice also that $\mathbf{T}_{\alpha}(N) = \mathbf{T}_{\alpha}(R*G \otimes_R K) \cap N$ for all α , thus,

$$\begin{aligned} \frac{\mathbf{T}_{\alpha+1}(N)}{\mathbf{T}_{\alpha}(N)} &= \frac{\mathbf{T}_{\alpha+1}(R*G \otimes_R K) \cap N}{\mathbf{T}_{\alpha}(R*G \otimes_R K) \cap N} \cong \\ &\cong \frac{(\mathbf{T}_{\alpha+1}(R*G \otimes_R K) \cap N) + \mathbf{T}_{\alpha}(R*G \otimes_R K)}{\mathbf{T}_{\alpha}(R*G \otimes_R K)} \leq \frac{\mathbf{T}_{\alpha+1}(R*G \otimes_R K)}{\mathbf{T}_{\alpha}(R*G \otimes_R K)} \end{aligned}$$

is different from zero for finitely many ordinals α . Thus,

$$\text{G.dim}(N) = \sup\{\alpha + 1 : \mathbf{T}_{\alpha+1}(N)/\mathbf{T}_{\alpha}(N) \neq 0\} = \max\{\alpha + 1 : \mathbf{T}_{\alpha+1}(N)/\mathbf{T}_{\alpha}(N) \neq 0\}$$

is clearly a successor ordinal. \square

Now, suppose that $\text{Ker}(\phi) \neq 0$ and let $\text{G.dim}(\text{Ker}(\phi)) = \alpha + 1$. We want to show that

$$\phi \upharpoonright_{\mathbf{T}_{\alpha+1}(M)} : \mathbf{T}_{\alpha+1}(M) \rightarrow \mathbf{T}_{\alpha+1}(M)$$

is surjective. Indeed, if there is $x \in \mathbf{T}_{\alpha+1}(M) \setminus \phi(\mathbf{T}_{\alpha+1}(M))$, it means that there exists $y \in M \setminus \mathbf{T}_{\alpha+1}(M)$ such that $\phi(y) = x$ (by the surjectivity of ϕ). This is to say that there is a short exact sequence

$$0 \rightarrow \text{Ker}(\phi) \cap R*Gy \rightarrow R*Gy \rightarrow R*Gx \rightarrow 0,$$

with $\text{G.dim}_R(R*Gy) \geq \alpha + 1 \geq \max\{\text{G.dim}_R(\text{Ker}(\phi) \cap R*Gy), \text{G.dim}_R(R*Gx)\}$, which contradicts Lemma 2.14(4). Thus, we have a short exact sequence of left $R*G$ -modules

$$0 \rightarrow \text{Ker}(\phi) \rightarrow \mathbf{T}_{\alpha+1}(M) \rightarrow \mathbf{T}_{\alpha+1}(M) \rightarrow 0.$$

Consider the length function $\ell_\alpha : R\text{-Mod} \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$ described in Subsection 3.3 and recall that $\text{Ker}(\ell_\alpha)$ is exactly the class of all left R -modules with Gabriel dimension $\leq \alpha$. Furthermore, $\mathbf{T}_{\alpha+1}(K)$ is a Noetherian module, thus, $\mathbf{Q}_\alpha(\mathbf{T}_{\alpha+1}(K))$ is a Noetherian object in a semi-Artinian category, that is, an object with finite composition length, for this reason $\ell_\alpha(\mathbf{T}_{\alpha+1}(K)) = \ell(\mathbf{Q}_\alpha(\mathbf{T}_{\alpha+1}(K))) < \infty$. Using the computations of Example 6.5 and the Addition Theorem, we get

$$\text{ent}_{\ell_\alpha}(\mathbf{T}_{\alpha+1}(R*G \otimes K)) = \ell_\alpha(\mathbf{T}_{\alpha+1}(K)) < \infty \quad \text{and} \quad \text{ent}_{\ell_\alpha}(\mathbf{T}_{\alpha+1}(M)) = \text{ent}_{\ell_\alpha}(\mathbf{T}_{\alpha+1}(M)) + \text{ent}_{\ell_\alpha}(\text{Ker}(\phi)).$$

Hence, $\text{ent}_{\ell_\alpha}(\text{Ker}(\phi)) = 0$ which, by Proposition 6.8, is equivalent to say that $\text{Ker}(\phi) \subseteq \text{Ker}(\ell_\alpha)$, contradicting the fact that $\text{G.dim}(\text{Ker}(\phi)) = \alpha + 1$. \square

In the above proof we made use of the Addition Theorem for the algebraic entropy, which is quite a deep result. We want to underline that if one is only interested in the second part of the statement, that is, stable finiteness of endomorphism rings, then it is sufficient to use the weaker additivity of the algebraic entropy on direct sums, which can be verified as an easy exercise independently from the Addition Theorem.

Example 7.11. Let G be a free group of rank ≥ 2 and let \mathbb{K} be a field. It is well-known that $\mathbb{K}[G]$ is not left (nor right) Noetherian so we can find a left ideal ${}_{\mathbb{K}[G]}I \leq \mathbb{K}[G]$ which is not finitely generated. Furthermore, by [6, Corollary 7.11.8], $\mathbb{K}[G]$ is a free ideal ring, so I is free. This means that I is isomorphic to a coproduct of the form $\mathbb{K}[G]^{(\mathbb{N})}$ which is obviously not hopfian.

Let us conclude this subsection with the following problem:

Problem 7.12. *Study the class of finitely generated groups G such that the group algebra $\mathbb{K}[G]$ is hereditarily Hopfian for any skew field \mathbb{K} . Is it true that this property characterizes the class of finitely generated amenable groups?*

One could also state an analogous problem including all the possible crossed products $\mathbb{K}*G$, instead of just the group algebras $\mathbb{K}[G]$.

7.2 Zero-Divisors

In this last section of the paper we discuss another classical conjecture due to Kaplansky about group rings connecting it to the theory of algebraic entropy:

Conjecture 7.13 (Kaplansky). *Let \mathbb{K} be a field and G be a torsion-free group. Then $\mathbb{K}[G]$ is a domain.*

Some cases of the above conjecture are known to be true but the conjecture is fairly open in general (for a classical reference on this conjecture see for example [28]). In most of the known cases, the strategy for the proof is to find an immersion of $\mathbb{K}[G]$ in some division ring. This is clearly sufficient but, in principle, it is a stronger property. To the best of the author's knowledge, the following question remains open: *Is it true that $\mathbb{K}[G]$ is a domain if and only if $\mathbb{K}[G]$ is a subring of a division ring?*

The above question is known to have positive answer if G is amenable (see [23, Example 8.16]). In the present section we provide an alternative argument to answer the above question for amenable groups (in the more general setting of crossed group rings) and we translate the amenable case of Conjecture 7.13 into an equivalent statement about algebraic entropy. This approach is inspired to the work of Nhan-Phu Chung and Andreas Thom [5]. Indeed, we can prove the following

Theorem 7.14. *Let \mathbb{K} be a division ring and let G be a finitely generated amenable group. For any fixed crossed product $\mathbb{K}*G$, the following are equivalent:*

- (1) $\mathbb{K}*G$ is a left (and right) Ore domain;
- (2) $\mathbb{K}*G$ is a domain;
- (3) $\text{ent}_{\dim}(\mathbb{K}*G M) = 0$, for every proper quotient M of $\mathbb{K}*G$;

$$(4) \text{ Im}(\text{ent}_{\dim}) = \mathbb{N} \cup \{\infty\}.$$

Before proving the above theorem we recall some useful properties about Ore domains. We start recalling that a domain D is left Ore if $Dx \cap Dy \neq \{0\}$ for all $x, y \in D \setminus \{0\}$. It can be shown that this is equivalent to say that D is a left flat subring of a division ring.

Proposition 7.15. *A domain D is left Ore if and only if there is a length function $L : D\text{-Mod} \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$ such that $L(D) = 1$.*

Proof. If D is left Ore, then D is a flat subring of a division ring \mathbb{K} . Then there is an exact functor $\mathbb{K} \otimes_D - : D\text{-Mod} \rightarrow \mathbb{K}\text{-Mod}$ which commutes with direct limits. Thus, we can define the desired length function L simply letting $L({}_D M) = \dim_{\mathbb{K}}(\mathbb{K} \otimes_D M)$.

On the other hand, suppose that there is a length function $L : D\text{-Mod} \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$ such that $L(D) = 1$ and choose $x, y \in D \setminus \{0\}$. Since D is a domain, both Dx and Dy contain (and are contained in) a copy of D , thus $L(Dx) = L(Dy) = 1$. If, looking for a contradiction $Dx \cap Dy = \{0\}$, then

$$1 = L(D) \geq L(Dx + Dy) = L(Dx \oplus Dy) = L(Dx) + L(Dy) = 2,$$

which is a contradiction. \square

It is a classical result that any left Noetherian domain is left Ore (see for example [24, Theorem 1.15 in Chapter 2.1]). By the above proposition we can generalize this result as follows:

Corollary 7.16. *A domain with left Gabriel dimension is necessarily left Ore.*

Proof. Let D be a domain with left Gabriel dimension. First of all we verify that $\text{G.dim}({}_D D)$ is not a limit ordinal. Indeed, if $\text{G.dim}({}_D D) = \lambda$ is a limit ordinal, then $D = \bigcup_{\alpha < \lambda} \mathbf{T}_{\alpha}(D)$. This means that, for any non-zero $x \in D$, there exists $\alpha < \lambda$ such that $Dx \in \mathbf{T}_{\alpha}(D)$. Choose a non-zero $x \in D$, as D is a domain, there is a copy of D inside Dx . Thus, $\text{G.dim}(D) \leq \text{G.dim}(Dx) \leq \alpha$ for some $\alpha < \lambda$, a contradiction.

If $\text{G.dim}({}_D D) = \alpha + 1$ for some ordinal α , then we can consider the length function

$$\ell_{\alpha} : D\text{-Mod} \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}, \quad \ell_{\alpha}(M) = \ell(\mathbf{Q}_{\alpha}(M)).$$

To conclude one has to show that $\ell_{\alpha}(D) = 1$, that is, $\mathbf{Q}_{\alpha}(D)$ is a simple object. Since $\mathfrak{C}_{\alpha+1}/\mathfrak{C}_{\alpha}$ is semi-Artinian, there is a simple subobject S of $\mathbf{Q}_{\alpha}(D)$. Then $\mathbf{S}_{\alpha}(S)$ is a sub-module of $\mathbf{S}_{\alpha}\mathbf{Q}_{\alpha}(D)$. Identify $\mathbf{S}_{\alpha}(S)$, $\mathbf{S}_{\alpha}\mathbf{Q}_{\alpha}(D)$ and D with submodules of $E(D)$, since D is essential in $E(D)$, there is $0 \neq x$ such that $x \in \mathbf{S}_{\alpha}(S) \cap D$, but then $\mathbf{S}_{\alpha}(S)$ contains an isomorphic copy of D . Thus $\mathbf{Q}_{\alpha}\mathbf{S}_{\alpha}(S) = S$ contains an isomorphic copy of $\mathbf{Q}_{\alpha}(D)$, which is therefore simple. \square

We can finally prove our result:

Proof of Theorem 7.14. (1) \Rightarrow (2) is trivial while (2) \Rightarrow (1) follows by Proposition 7.15 and the fact that the algebraic dim-entropy is a length function on $\mathbb{K}*G\text{-Mod}$ such that $\text{ent}_{\dim}(\mathbb{K}*G) = 1$.

(2) \Rightarrow (3). Consider a short exact sequence $0 \rightarrow \mathbb{K}*G I \rightarrow \mathbb{K}*G \mathbb{K}*G \rightarrow \mathbb{K}*G M \rightarrow 0$, with $I \neq 0$. Choose $0 \neq x \in I$, then $\mathbb{K}*G x \cong \mathbb{K}*G$, and so $\text{ent}_{\dim}(\mathbb{K}*G M) = \text{ent}_{\dim}(\mathbb{K}*G \mathbb{K}*G) - \text{ent}_{\dim}(\mathbb{K}*G I) \leq 1 - 1 = 0$.

(3) \Rightarrow (4). Let us show first that for any finitely generate left $\mathbb{K}*G$ -module $\mathbb{K}*G F$, $\text{ent}_{\dim}(\mathbb{K}*G F) \in \mathbb{N}$. In fact, choose a finite set of generators x_1, \dots, x_n for F and, letting $F_0 = 0$ and $F_i = \mathbb{K}*G x_1 + \dots + \mathbb{K}*G x_i$ for all $i = 1, \dots, n$, consider the filtration $0 \subseteq F_1 \subseteq F_2 \subseteq \dots \subseteq F_n = F$. By additivity,

$$\text{ent}_{\dim}(F) = \sum_{i=1}^n \text{ent}_{\dim}(F_i/F_{i-1}).$$

All the modules $F_1/F_0, \dots, F_n/F_{n-1}$ are cyclics (i.e. quotients of $\mathbb{K}*G$), thus $\text{ent}_{\dim}(F_i/F_{i-1}) \in \{0, 1\}$ by hypothesis. Hence, $\text{ent}_{\dim}(F) \in \mathbb{N}$. To conclude one argues by upper continuity that the algebraic dim-entropy of an arbitrary left $\mathbb{K}*G$ -module is the supremum of a subset of \mathbb{N} , thus it belongs to $\mathbb{N} \cup \{\infty\}$.

(4) \Rightarrow (2). Let $x \in \mathbb{K}*G$ and consider the short exact sequence

$$0 \rightarrow I \rightarrow \mathbb{K}*G \rightarrow \mathbb{K}*G x \rightarrow 0$$

where $I = \{y \in \mathbb{K}*G : yx = 0\}$. Suppose that x is a zero-divisor, that is, $I \neq 0$ or, equivalently, $\dim({}_{\mathbb{K}} I) \neq 0$. By Proposition 6.8, $\text{ent}_{\dim}(I) > 0$ and, by our assumption, $\text{ent}_L(I) \geq 1$. Hence, using additivity, $\text{ent}_{\dim}(\mathbb{K}*G x) = 0$. Again by Proposition 6.8, this implies $\dim(\mathbb{K}*G x) = 0$ and consequently $\mathbb{K}*G x = 0$, that is, $x = 0$. Thus, the unique zero-divisor in $\mathbb{K}*G$ is 0. \square

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